DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCE-MATHEMATICS

SEMESTER –II

COMPLEX ANALYSIS-II

DEMATH-2 ELEC-5

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

COMPLEX ANALYSIS II

BLOCK-1

BLOCK-2

BLOCK-1 COMPLEX ANALYSIS II

Introduction to block

Unit 1 Harmonic Function : This unit deals with harmonic function and Riemann Sheet

Unit 2 Analytic And Harmonic Function : Deals with analytic function and harmonic function with its examples.

Unit 3 Application of Harmonic Function : Deals with harmonic function , Poisson Integral Formula and its proof

Unit 4 The Dirichlet Problem for the Unit Disk and Fourier Series : Deals with Fourier Cosine series and Dirichlet Problem for the unit disk, also deals with Polar form of a complex number

Unit 5 Geometric Series and Convergence : Deals with Zeno's Paradoxes and Operation on convergence series. Also deals with sequence and series

Unit 6 Principal of Convergence : Deals with Cauchy Criterian and its examples. Deals Weierstrass Product Inequality

Unit 7 Convergence of Infinite Product : Deals with infinite product and its examples. Also deals with Uniform convergence and Weierstrass M-Test

UNIT 1: HARMONIC FUNCTION

STRUCTURE

1.0 OBJECTIVE

In this part of the course, we will study some basic complex analysis. This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematics and physics. We will extend the notions of derivatives and integrals, familiar from calculus to the case of complex functions of a complex variable. In so doing we will come across analytic functions, which form the centerpiece of this part of the course. In fact, to a large extent complex analysis is the study of analytic functions. After a brief review of complex numbers as points in the complex plane, we will first discuss analyticity and give plenty of examples of analytic functions. We will then discuss complex integration, culminating with the generalized Cauchy Integral Formula, and some of its applications. We then go on to discuss the power series representations of analytic functions and the residue calculus, which will allow us to compute many real integrals and infinite sums very easily via complex integration.

1.1 INTRODUCTION

Definition: A real-valued function $\emptyset(x,y)$is harmonic in a domain D if all of its second partials are continuous in D and if at each point in D , $\emptyset$ is analytic in a domain D, then both u(x,y), v(x,y) are harmonic in D

Definition: A complex-valued function F(z) is holomorphic on an open set G if it has a derivative at every point in G.

Here, Holomorphicity is defined over an open set, however, differentiability could only at one point. If f(z) is holomorphic over the entire complex plane, we say that f is *entire*. As an example, all polynomial functions of z are entire.

1.2 HARMONIC FUNCTIONS AND THEIR RIEMANN SHEETS

Let $\phi(x, y)$ be a continuous real-valued function of the two real variables x and y that is defined on a domain D. A domain D is a connected and open set of points in the complex plane.) The partial differential equation $\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$, is known as Laplace's equation and is sometimes referred to as the potential equation.

If

$\phi_x(x, y)$, $\phi_y(x, y)$, $\phi_{xx}(x, y)$, $\phi_{xy}(x, y)$, $\phi_{yx}(x, y)$ and $\phi_{yy}(x, y)$ are all continuous, and if $\phi(x, y)$ satisfies Laplace's equation, then

$\phi(x, y)$ is called a harmonic function. In calculus, we might have been asked to show that polynomial functions like

$u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$, and

transcendental functions like $u(x, y) = e^x \cos(y)$ and

$v(x, y) = e^x \sin(y)$, and $u(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$ and

$v(x, y) = \arctan\left(\frac{y}{x}\right)$, are all harmonic functions. These pairs of functions are not chosen at random, and there is an intimate relationship between them, they are called the conjugate "harmonic functions." It is our goal to understand how this concept is tied in with analytic functions.

On the practical side, harmonic functions are important in the areas of applied mathematics, engineering, and mathematical physics. Harmonic functions are used to solve problems involving steady-state temperatures, two-dimensional electrostatics, and ideal fluid flow. we will show how complex analysis techniques are used to solve these problems. For example, the function

$\phi(x, y) = \frac{1}{\pi}\arctan\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\arctan\left(\frac{y}{x+1}\right)$, is harmonic in the upper half-plane and takes on the boundary values

$\phi(x, 0) = 1$ when $|x| < 1$ and $\phi(x, 0) = 0$ when $|x| > 1$.

harmonic

function $\phi(x, y) = \frac{1}{\pi}\arctan\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\arctan\left(\frac{y}{x+1}\right)$.

We begin with an important theorem relating analytic and harmonic functions.

Theorem 3.1. Let $f(z) = f(x + iy) = u(x, y) + i v(x, y)$ be an analytic function on a domain D. Then both $u(x, y)$ and $v(x, y)$ are harmonic functions on D. In other words, the real and imaginary parts of an analytic function are harmonic.

Notes

Proof. Since $f(z)$ is differentiable on D, the Cauchy-Riemann equations that $u_x(x, y) = v_y(x, y)$ and $u_y(x, y) = -v_x(x, y)$, and that $f'(z) = u_x(x, y) + i v_x(x, y) = v_y(x, y) - i u_y(x, y)$. we will prove that if $f(z)$ is analytic on D, then $f'(z)$ is also analytic on D. Since $f'(z)$ is differentiable on D, the Cauchy-Riemann equations imply that all the second partial derivatives: $u_{xx}(x, y)$, $u_{xy}(x, y)$, $u_{yy}(x, y)$ and $v_{xx}(x, y)$, $v_{xy}(x, y)$, $v_{yy}(x, y)$, exist are and are continuous on D.

Using these facts, we can start with the above mentioned Cauchy Riemann equations and take the partial derivative with respect to x of each side of these equations and obtain $u_{xx}(x, y) = v_{yx}(x, y)$ and $u_{yx}(x, y) = -v_{xx}(x, y)$. Similarly, taking the partial derivative of each side with respect to y yields $u_{xy}(x, y) = v_{yy}(x, y)$ and $u_{yy}(x, y) = -v_{xy}(x, y)$. Since the partial derivatives $u_{xy}(x, y)$, $u_{yx}(x, y)$, $v_{xy}(x, y)$, and $v_{yx}(x, y)$ are all continuous, we use a theorem from the calculus of real functions that states that the mixed partial derivatives are equal; that is, $u_{xy}(x, y) = u_{yx}(x, y)$ and $v_{xy}(x, y) = v_{yx}(x, y)$. Combining all these results finally gives $u_{xx}(x, y) + u_{yy}(x, y) = v_{yx}(x, y) - v_{xy}(x, y) = 0$, and $v_{xx}(x, y) + v_{yy}(x, y) = -u_{yx}(x, y) + u_{xy}(x, y) = 0$. Therefore both $u(x, y)$ and $v(x, y)$ are harmonic functions on D.

Definition (Harmonic Conjugate). If we have a function $u(x, y)$ that is harmonic on the domain D and if we can find another harmonic function $v(x, y)$ such that the partial derivatives for $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations throughout D, then we say that $v(x, y)$ is a harmonic conjugate of $u(x, y)$. Furthermore, it then follows that the function $f(z) = f(x + iy) = u(x, y) + i v(x, y)$ is analytic on D.

The unlocks the relationship among harmonic functions, conjugate harmonic functions and analytic functions. Specifically, it clearly states the special relationship between a harmonic function and it's conjugate harmonic function. Loosely speaking, the harmonic function is the real part of the given analytic function and the harmonic conjugate function is the imaginary part of the given analytic function.

Example 3.2 Show that $u(x, y) = x^2 - y^2$ is a harmonic function and find a conjugate harmonic function $v(x, y)$, and an analytic function $f(z) = f(x + i y) = u(x, y) + i v(x, y)$.

Solution. Given $u(x, y) = x^2 - y^2$, we have $u_x(x, y) = 2x$ and $u_y(x, y) = 2y$ and the second partial derivatives are $u_{xx}(x, y) = 2$ and $u_{yy}(x, y) = -2$. It follows that $u_{xx}(x, y) + u_{yy}(x, y) = 2 - 2 = 0$, hence $u(x, y) = x^2 - y^2$ is a harmonic function for all $z = x + i y$.

If we choose $v(x, y) = 2xy$, we have $v_x(x, y) = 2y$ and $v_y(x, y) = 2x$ and the second partial derivatives are $v_{xx}(x, y) = 0$ and $v_{yy}(x, y) = 0$. It follows that $v_{xx}(x, y) + v_{yy}(x, y) = 0 - 0 = 0$, hence $v(x, y) = 2xy$ is a harmonic function for all $z = x + i y$.

Therefore, the harmonic conjugate of $u(x, y) = x^2 - y^2$, is $v(x, y) = 2xy$.

Furthermore, u and v satisfy the Cauchy-Riemann equations $u_x(x, y) = v_y(x, y) = 2x$, and $u_y(x, y) = -v_x(x, y) = -2y$. Therefore, $f(z) = f(x + i y) = x^2 - y^2 + i 2xy$ is an analytic function.

Alternative Solution. The function $f(z) = z^2 = (x + i y)^2 = x^2 - y^2 + i 2xy$ is analytic for all values of z. Hence, it follows from that both $u(x, y) = \mathrm{Re}[f(z)] = x^2 - y^2$, and $v(x, y) = \mathrm{Im}[f(z)] = 2xy$, are harmonic functions.

Example 3.3 Show that $v(x, y) = 3x^2 y - y^3$ is a harmonic conjugate of $u(x, y) = x^3 - 3xy^2$.

Notes

Solution. Given $u(x, y) = x^3 - 3xy^2$, we have $u_x(x, y) = 3x^2 - 3y^2$ and $u_y(x, y) = -6xy$ and the second partial derivatives are $u_{xx}(x, y) = 6x$ and $u_{yy}(x, y) = -6x$. It follows that $u_{xx}(x, y) + u_{yy}(x, y) = 6x - 6x = 0$, hence $u(x, y) = x^3 - 3xy^2$ is a harmonic function for all $z = x + iy$.

Similarly, for $v(x, y) = 3x^2 y - y^3$, we have $v_x(x, y) = 6xy$ and $v_y(x, y) = 3x^2 - 3y^2$ and the second partial derivatives are $v_{xx}(x, y) = 6y$ and $v_{yy}(x, y) = -6y$. It follows that $v_{xx}(x, y) + v_{yy}(x, y) = 6y - 6y = 0$, hence $v(x, y) = 3x^2 y - y^3$ is a harmonic function for all $z = x + iy$.

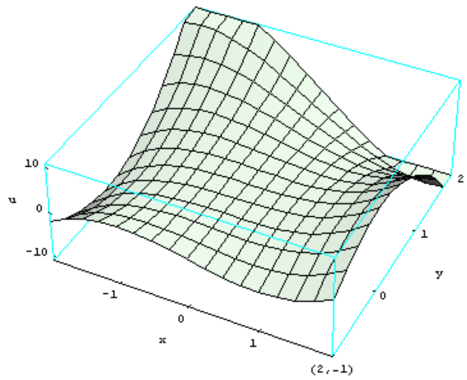
Furthermore, u and v satisfy the Cauchy-Riemann equations $u_x(x, y) = v_y(x, y) = 3x^2 - 3y^2$, and $u_y(x, y) = -v_x(x, y) = -6xy$. we see that $f(z) = f(x + iy) = (x^3 - 3xy^2) + i(3x^2 y - y^3)$ is an analytic function.

Therefore, the harmonic conjugate of $u(x, y) = x^3 - 3xy^2$, is $v(x, y) = 3x^2 y - y^3$.

Alternative Solution. The function $f(z) = z^3 = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2 y - y^3)$ is analytic for all values of z. Hence, it follows that both $u(x, y) = \mathrm{Re}[f(z)] = x^3 - 3xy^2$, and $v(x, y) = \mathrm{Im}[f(z)] = 3x^2 y - y^3$, are harmonic functions.

Therefore, the harmonic conjugate of $u(x, y) = x^3 - 3xy^2$, is $v(x, y) = 3x^2 y - y^3$.

Aside. $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2 y - y^3$. The partial derivatives of $u(x, y)$ are $u_x(x, y) = 3x^2 - 3y^2$ and $u_y(x, y) = -6xy$, and the partial derivatives of $v(x, y)$ are $v_x(x, y) = 6xy$ and $v_y(x, y) = 3x^2 - 3y^2$. They satisfy the Cauchy-Riemann equations because they are the real and imaginary parts of an analytic function. At the point $(x, y) = (2, -1)$, we have $u_x(2,-1) = 9$ and $v_y(2,-1) = 9$, and these partial derivatives appear along the edges of the surfaces for $u(x,y)$ and $v(x,y)$ at the points $(2,-1,u(2,-1))$ and $(2,-1,v(2,-1))$, respectively. Similarly, at the point $(x,y) = (2,-1)$, we have $u_y(2,-1) = 12$ and $v_x(2,-1) = -12$ and these partial derivatives appear along the edges of the surfaces for $u(x,y)$ and $v(x,y)$ at the points $(2,-1,u(2,-1))$ and $(2,-1,v(2,-1))$, respectively.



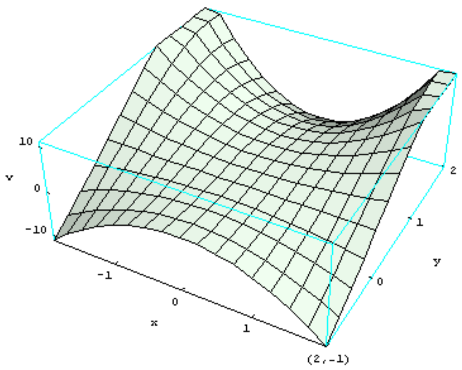
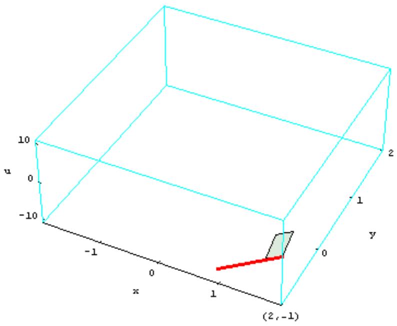


Figure 3.2 a $u(x,y) = x^3 - 3xy^2$. **Figure 3.3 a** $v(x,y) = 3x^2y - y^3$.



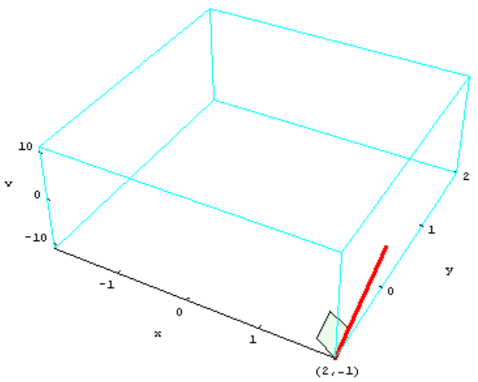
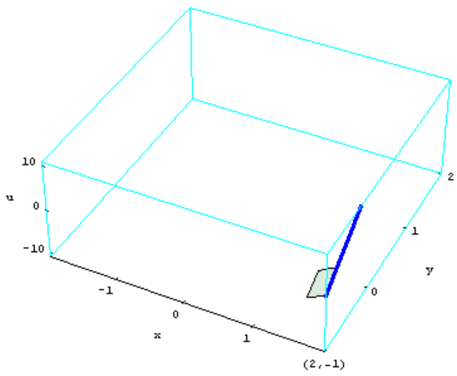


Figure 3.2 b $u(x,y) = x^3 - 3xy^2$, at $(2,-1)$ we have $u_x(2,-1) = 9$. **Figure 3.3 b** $v(x,y) = 3x^2y - y^3$, at $(2,-1)$ we have $v_y(2,-1) = 9$.



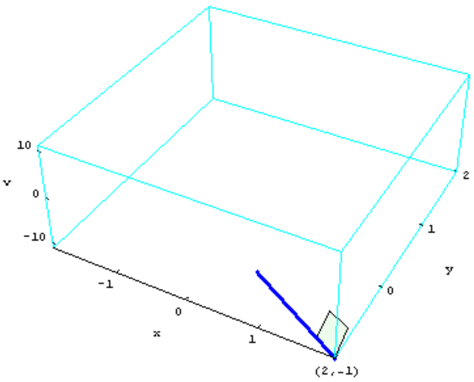


Figure 3.2 c $u(x, y) = x^3 - 3xy^2$, **Figure 3.3 c** $v(x, y) = 3x^2 y - y^3$, at $(2, -1)$ we have $u_y(2, -1) = 12$.

For the function $f(z) = f(x + i y) = x^3 - 3xy^2 + i(3x^2 y - y^3)$ we see that $u_x(2, -1) = 9 = v_y(2, -1)$ and $u_y(2, -1) = 12 = -v_x(2, -1)$.

A question about the harmonic conjugate.

If $v(x, y)$ is the harmonic conjugate of $u(x, y)$, then is $v(x, y)$ is the harmonic conjugate of $u(x, y)$? The following example shows that this is not the case, and $v(x, y)$ is not the harmonic conjugate of $u(x, y)$.

Extra Example 3.4. Given the harmonic functions $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2 y - y^3$, and the analytic function $f(z) = f(x + i y) = u(x, y) + i v(x, y)$.

3.4 (a) Show that $g(z) = g(x + i y) = v(x, y) + i u(x, y)$ is not an analytic function.

Solution. We can write

$$g(z) = g(x + i y)$$

$$= v(x, y) + i u(x, y)$$

$$= (3x^2 y - y^3) + i(x^3 - 3xy^2)$$

Now $g(z)$ can be expressed in the form

$g(z) = g(x+iy) = U(x,y) + iV(x,y)$ where $U(x,y) = 3x^2y - y^3$,

and $V(x,y) = x^3 - 3xy^2$

The partial derivatives of $U(x,y)$ and $V(x,y)$ are $U_x(x,y) = 6xy$ and $U_y(x,y) = 3x^2 - 3y^2$, and $V_x(x,y) = 3x^2 - 3y^2$ and $V_y(x,y) = -6xy$. Now check out the the Cauchy-Riemann equations

$U_x(x,y) = 6xy \neq -6xy = V_y(x,y)$,

and $U_y(x,y) = 3x^2 - 3y^2 \neq -(3x^2 - 3y^2) = -V_x(x,y)$. The Cauchy-Riemann equations hold only at the isolated point $(x,y) = (0,0)$. Therefore, $f(z) = f(x+iy) = (3x^2y - y^3) - i(x^3 - 3xy^2)$ is not an analytic function.

We are done.

3.4 (b) Show that $h(z) = h(x+iy) = v(x,y) - iu(x,y)$ is an analytic function, for all z.

Solution. We can write

$$h(z) = h(x+iy)$$

$$= v(x,y) - iu(x,y)$$

$$= (3x^2y - y^3) - i(x^3 - 3xy^2)$$

$$= (3x^2y - y^3) + i(-x^3 + 3xy^2)$$

Now $h(z)$ can be expressed in the form

$h(z) = h(x+iy) = U(x,y) + iV(x,y)$ where $U(x,y) = 3x^2y - y^3$

and $V(x,y) = -x^3 + 3xy^2$

The partial derivatives of U and V are $U_x(x,y) = 6xy$ and $U_y(x,y) = 3x^2 - 3y^2$, and $V_x(x,y) = -3x^2 + 3y^2$ and $V_y(x,y) = 6xy$. Now check out the Cauchy-Riemann equations

$U_x(x,y) = 6xy = V_y(x,y)$,

and $U_y(x,y) = 3x^2 - 3y^2 = -(-3x^2 + 3y^2) = -V_x(x,y)$. we see

Notes

that $h(z) = h(x+iy) = (3x^2y-y^3) - i(x^3-3xy^2)$ is an analytic function, for all z.

We can use complex analysis to show easily that certain combinations of harmonic functions are harmonic. For example, if $v(x,y)$ is a harmonic conjugate of $u(x,y)$, then their product $u(x,y)v(x,y)$ is a harmonic function. This can be verified directly by computing the partial derivatives and showing that Laplace's equation (3-26) holds, but the details are tedious. If we use complex variable techniques instead, we can start with the fact that $f(z) = u(x,y) + iv(x,y)$ is an analytic function. Then we observe that the square of $f(z)$ is also an analytic function, which is $(f(z))^2 = \phi(x,y) + i\psi(x,y)$, which can be written as $(f(z))^2 = (u(x,y))^2 - (v(x,y))^2 + i2u(x,y)v(x,y)$. We then know immediately that the imaginary part, $\psi(x,y) = 2u(x,y)v(x,y)$, is a harmonic function by Theorem 3.8. Since a constant multiple of a harmonic function is harmonic, it follows that $u(x,y)v(x,y)$ is harmonic. It is left as an exercise to show that if $u_1(x,y)$ and $u_2(x,y)$ are two harmonic functions that are not related in the preceding fashion, then their product need not be harmonic.

Method I. Construction of the Harmonic Conjugate of u(x,y) using Integration.

We now introduce methods for the construction of a harmonic conjugate function. The first method uses familiar techniques of calculus.

Check-in Progress-1

Note: Please give a solution of questions in space gives below:

Q. 1 Define Harmonic Conjugate

Solution :

……………………………………………………………………………

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……………………………………………………………………………

Q. 2 Give Definition of Harmonic Function.

Solution :

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Theorem 3.5 (Construction of a Conjugate). Let $u(x, y)$ be harmonic in an ϵ-neighborhood of the point (x_0, y_0). Then there exists a conjugate harmonic function $v(x, y)$ defined in this neighborhood such that $f(z) = u(x, y) + i v(x, y)$, is an analytic function.

Proof. A conjugate harmonic function $v(x, y)$ will satisfy the Cauchy-Riemann equations $u_x(x, y) = v_y(x, y)$ and $u_y(x, y) = -v_x(x, y)$. Assuming that such a function exists, we determine what it would have to look like by using a two-step process. First, we integrate $v_y(x, y)$ (which should equal $u_x(x, y)$) with respect to y and

$$v(x, y) = \int v_y(x, y)\, dy + C(x)$$

get (3-27) $v(x, y) = \int u_x(x, y)\, dy + C(x)$ where $C(x)$ is a function of x alone that is yet to be determined. Second, we compute $C'(x)$ by differentiating both sides of this equation with respect to x and replacing $v_x(x, y)$ with $-u_y(x, y)$ on the left side, which gives

$$v_x(x, y) = \frac{d}{dx} \int u_x(x, y)\, dy + C'(x)$$

$$-u_y(x, y) = \frac{d}{dx} \int u_x(x, y)\, dy + C'(x)$$

It can be shown (we leave the details for the reader) that because u is harmonic, all terms except those involving x in the last equation will cancel, revealing a formula for $C'(x)$ involving x alone. Elementary integration of the single-variable function $C'(x)$ can then be used to discover $C(x)$. We finally observe that the function $v(x, y)$ so created indeed has the properties we seek.

Notes

The functions $C(x)$ and $v(x, y)$ are computed with the formulas:

$$C(x) = \int \left(-u_y(x, y) - \frac{d}{dx}\left(\int u_x(x, y)\, dy \right) \right) dx$$

, and $$v(x, y) = \int u_x(x, y)\, dy + C(x).$$

Remark. If you prefer a more succinct formula, then the harmonic conjugate of $u(x, y)$ is given by

$$v(x, y) = \int u_x(x, y)\, dy - \int u_y(x, y)\, dx - \int\int u_{xx}(x, y)\, dy\, dx.$$

Proof. Technically we should always specify the domain of a function when we define it. When no such specification is given, it is often assumed that the domain is the entire complex plane, or the largest set for which the expression defining the function which makes sense.

Example 3.6 Show that $u(x, y) = x y^3 - x^3 y$ is a harmonic function and find the harmonic conjugate $v(x, y)$.

Solution. We follow the construction process. The first partial derivatives are (3-28) $u_x(x, y) = y^3 - 3x^2 y$ and $u_y(x, y) = 3x y^2 - x^3$. To verify that $u(x, y)$ is harmonic, we compute the second partial derivatives and note that $u_{xx}(x, y) + u_{yy}(x, y) = -6xy + 6xy = 0$, so $u(x, y)$ satisfies Laplace's Equation (3-26). To construct $v(x, y)$, we start with Equation (3-27) and the first of Equations (3-28) and the Cauchy-Riemann equation $v_y(x, y) = u_x(x, y)$ and get

$$v(x, y) = \int v_y(x, y)\, dy + C(x)$$

$$= \int u_x(x, y)\, dy + C(x)$$

$$= \int (y^3 - 3x^2 y)\, dy + C(x)$$

$$= \frac{1}{4} y^4 - \frac{3}{2} x^2 y^2 + C(x)$$

We now need to differentiate the left and right sides of this equation with respect to x, $v_x(x, y) = \frac{d}{dx}\left(\int u_x(x, y)\, dy \right) + C'(x)$. Use Equation and the Cauchy-Riemann equation $-u_y(x, y) = v_x(x, y)$ to obtain $-3xy^2 + x^3 = 0 - 3xy^2 + C'(x)$ It follows easily that $C'(x) = x^3$, then an easy integration yields $C(x) = \int x^3 dx = \frac{1}{4}x^4 + c$, where c is a real constant. For convenience, we can choose $c = 0$.

Therefore, $v(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 - \frac{3}{2}x^2y^2$.

The "ghost of the imaginary numbers" - the subtle connection between Harmonic and Analytic Functions.

When you look at a family of level curves of a real function $u(x, y)$, do you naturally think of complex numbers? Certainly, it not the first thing that pops into our minds. However, it seems to be a subtle fact when studying complex analysis.

We cannot fail to stress the importance of the harmonic function pair that is constructed with Theorem 3.4 and Theorem 3.5. The orthogonal grid formed by the families of harmonic functions and how complex functions are used to find them is one goal of this book and is discussed in detail in Chapter 11. In reality, they are constructed with inverse functions $z = f^{-1}(w)$. It will take a while to feel comfortable with these concepts and that is why they are studied later in the book. For the time being do not worry about them, they are merely ghosts of the imaginary numbers.

For practical purposes, it suffices to consider regions in the z-plane and their image in the w-plane. However, the concept of a Riemann surface as being a "two dimensional manifold" has been around for a long time. So it is no surprise that things get sticky. The reader can do research and see that work being done regarding harmonic functions on Riemann surfaces (and also on foliations).

1.2.1 Applications of Harmonic Functions

we will introduce the complex potential $F(z) = \phi(x, y) + i\psi(x, y)$, which is an analytic function and $\phi(x, y)$ and $\psi(x, y)$, are harmonic

Notes

functions. It has many physical interpretations, some of which are listed below.

Physical Phenomenon	$\phi(x, y)$ = constant	$\psi(x, y)$ = constant
Heat flow	Isothermals	Heat flow lines
Electrostatics	Equipotential curves	Flux lines
Fluid flow	Equipotentials	Streamlines
Gravitational field	Gravitational potential	Lines of force
Magnetism	Potential	Lines of force
Diffusion	Concentration	Lines of flow
Elasticity	Strain function	Stress lines
Current flow	Potential	Lines of flow

Interpretations for the level curves of $\phi(x, y)$ and $\psi(x, y)$.

We do not have time to explore all of these applications at this time. So we will introduce the topic of ideal fluid flow.

1.3 IDEAL FLUID FLOW

We assume that an incompressible and frictionless fluid flows over the complex plane and that all cross-sections in planes parallel to the complex plane are the same. Situations such as this occur when fluid is flowing in a deep channel. The velocity vector at the point (x, y) is (3-29) $\vec{V}(x, y) = p(x, y) + i\,q(x, y)$.

The assumption that the flow is irrational and has no sources or sinks implies that both the curl and divergence vanish, that is, (3-30) $q_x(x, y) - p_y(x, y) = 0$ and $p_x(x, y) + q_y(x, y) = 0$. Hence $p(x, y)$ and $q(x, y)$ obey the partial differential equations (3-30) $p_x(x, y) = -q_y(x, y)$, (3-30) and $p_y(x, y) = q_x(x, y)$.

Equations (3-30) are similar to the Cauchy-Riemann equations and permit us to define a special complex function: (3-31) $f(z) = u(x, y) + i\,v(x, y) = p(x, y) - i\,q(x, y)$. Here we have (3-

30) $u_x(x, y) = p_x(x, y)$, $u_y(x, y) = p_y(x, y)$, and (3-30) $v_x(x, y) = -q_x(x, y)$, $v_y(x, y) = -q_y(x, y)$ We can use Equations (3-30) to verify that the Cauchy-Riemann equations are satisfied for $f(x + i y)$: $u_x(x, y) = p_x(x, y) = -q_y(x, y) = v_y(x, y)$, and $u_y(x, y) = p_y(x, y) = q_x(x, y) = -v_x(x, y)$. Assuming the functions $p(x, y)$ and $q(x, y)$ have continuous partials, Theorem 3.4 guarantees that function $f(x + i y)$ defined in Equation (3-31) is analytic, and that the fluid flow of Equation (3-29) is the conjugate of an analytic function, that is, $\vec{V}(x, y) = \overline{f(z)}$.

In Section 6.4 we will prove that every analytic function $f(x + i y)$ has an analytic antiderivative $F(x + i y)$; assuming this to be the case, we can write (3-32) $F(x + i y) = \phi(x, y) + i \psi(x, y)$, (3-30) where (3-30) $F'(x + i y) = f(x + i y)$. Theorem 3.8 tells us that $\phi(x, y)$ is a harmonic function. If we use the vector interpretation of a complex number we see that the gradient of $\phi(x, y)$ can be written as $\text{grad}\,\phi(x, y) = \phi_x(x, y) + i \phi_y(x, y)$.

The Cauchy-Riemann equations applied to $F(x + i y)$ give $\phi_y(x, y) = -\psi_x(x, y)$; making this substitution in the last equation yields $\text{grad}\,\phi(x, y) = \phi_x(x, y) - i \psi_x(x, y) = \overline{\phi_x(x, y) + i \psi_x(x, y)}$. Equation (3-14) says that $\phi_x(x, y) + i \psi_x(x, y) = F'(x + i y)$, which by the preceding equation and Equation (3-32) imply that $\text{grad}\,\phi(x, y) = \overline{F'(x + i y)} = \overline{f(x + i y)}$. Finally, from Equation (3-29), $\phi(x, y)$ is the scalar potential function for the a fluid flow, so $\vec{V}(x, y) = \text{grad}\,\phi(x, y)$.

Definition. Given the complex potential $F(z) = \phi(x, y) + i \psi(x, y)$. The curves $\{(x, y) : \phi(x, y) = \text{constant}\}$ are called equipotentials, and the curves $\{(x, y) : \psi(x, y) = \text{constant}\}$ are called streamlines. They are used to describe the path of fluid flow.

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In Section 11.4 we will see that the family of equipotentials is orthogonal to the family of streamlines.

Example 3.14. Show that the harmonic function $\phi(x, y) = x^2 - y^2$ is the scalar potential function for the fluid flow $\vec{V}(x, y) = 2x - i2y$.

Solution. We can write the fluid flow expression as $\vec{V}(x, y) = 2x - i2y = \overline{2z}$. Then use the equation $\vec{V}(x, y) = \overline{f(z)}$. It is easy to see that an antiderivative of $f(z) = 2z$ is $F(z) = z^2$. Therefore, $F(z) = z^2$ is the complex potential The real part of $F(z)$ is the scalar potential function function: $\phi(x, y) = \mathrm{Re}[F(z)] = \mathrm{Re}[x^2 - y^2 + i2xy] = x^2 - y^2$. Note that the hyperbolas $\phi(x, y) = x^2 - y^2 = C$ are the equipotential curves, and that the hyperbolas $\psi(x, y) = 2xy = C$ are the streamline curves, these curves are orthogonal, as shown in Figure 3.6.

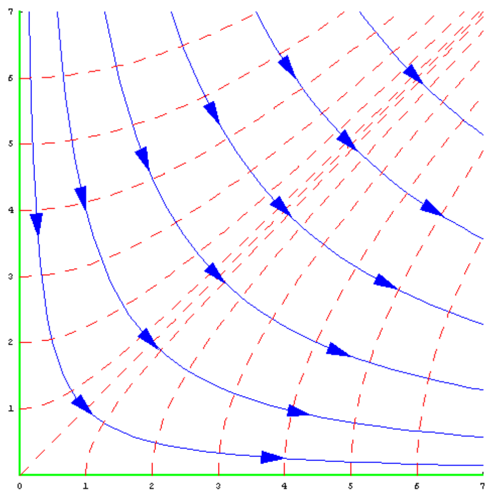


Figure 3.6 Red equipotential curves $\phi(x, y) = x^2 - y^2 = C$, and blue streamline curves $\psi(x, y) = 2xy = C$, for the complex potential $w = F(z) = z^2$.

Method II. Construction of the Harmonic Conjugate of u(x,y) using Algebra.

The usual method proposed for finding the harmonic conjugate uses integrals and derivatives and is shown above as Method I. A second method discovered by the British mathematician Louis Melville Milne-Thomson (1891-1974) uses novel algebraic construction. His method appears in the article On the Relation of an Analytic Function of z to Its Real and Imaginary Parts, L. M. Milne-Thomson, The Mathematical Gazette, Vol. 21, No. 244 (July 1937), pp. 228-229, Jstor. A good reference to read is the recent article, Recovering Holomorphic Functions from Their Real or Imaginary Parts without the Cauchy-Riemann Equations, William T. Shaw, SIAM Review, Vol 46, No. 4, 2004, pp 717-718, Jstor.

The Milne-Thomson Method for constructing a harmonic conjugate. **(i)** Given the harmonic function $u(x, y)$ then construct $v(x, y) = \mathrm{Im}\left[2u\left(\frac{x+iy}{2}, \frac{x+iy}{2i}\right)\right]$. Under the proper conditions, $v(x, y)$ is a harmonic conjugate of $u(x, y)$, and $f(x+iy) = u(x, y) + iv(x, y) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0)$ is an analytic function.

Proof of (i).

(ii) Given the harmonic function $v(x, y)$ then construct $u(x, y) = \mathrm{Re}\left[i2v\left(\frac{x+iy}{2}, \frac{x+iy}{2i}\right)\right]$. Under the proper conditions, $v(x, y)$ is a harmonic conjugate of $u(x, y)$, and $f(x+iy) = u(x, y) + iv(x, y) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) - iv(0, 0)$ is an analytic function.

1.4 LIMITATIONS OF THE MILNE-THOMSON METHOD

Observe that in Milne-Thomson method, the term $x^2 + y^2$ will be transformed into

Notes

$$\text{Im}\left[2\left(\left(\frac{x+i\,y}{2}\right)^2+\left(\frac{x+i\,y}{2\,i}\right)^2\right)\right] = \text{Im}\left[\frac{x^2}{2}+i\,x\,y-\frac{y^2}{2}-\frac{x^2}{2}-i\,x\,y+\frac{y^2}{2}\right]$$

$$= \text{Im}[0]$$

$$= 0$$

and that the term $\frac{x}{y}$ will be transformed

$$\text{Im}\left[2\,\frac{\frac{x+i\,y}{2}}{\frac{x+i\,y}{2\,i}}\right] = \text{Im}\left[2\,\frac{1}{\frac{1}{i}}\right]$$

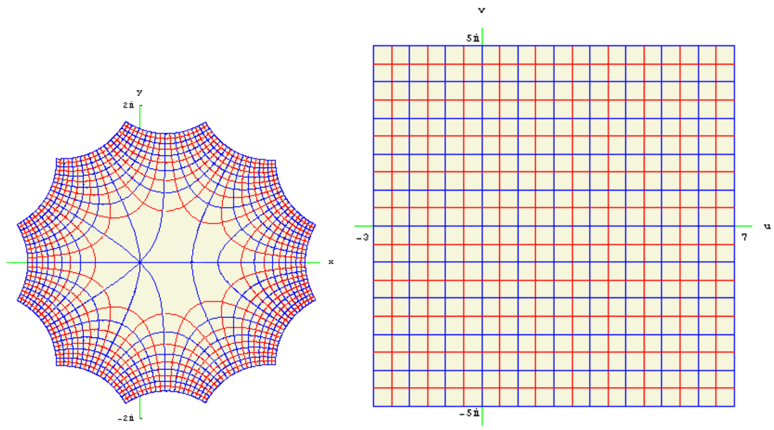
$$= \text{Im}[2\,i]$$

into $= 2$. Hence the method does not work if the given harmonic function contains a term that is the real or imaginary part

of $\text{Log}\,(x+i\,y)$, $\frac{1}{x+i\,y}$, or $\frac{1}{(x+i\,y)^n}$, where n is a positive integer.

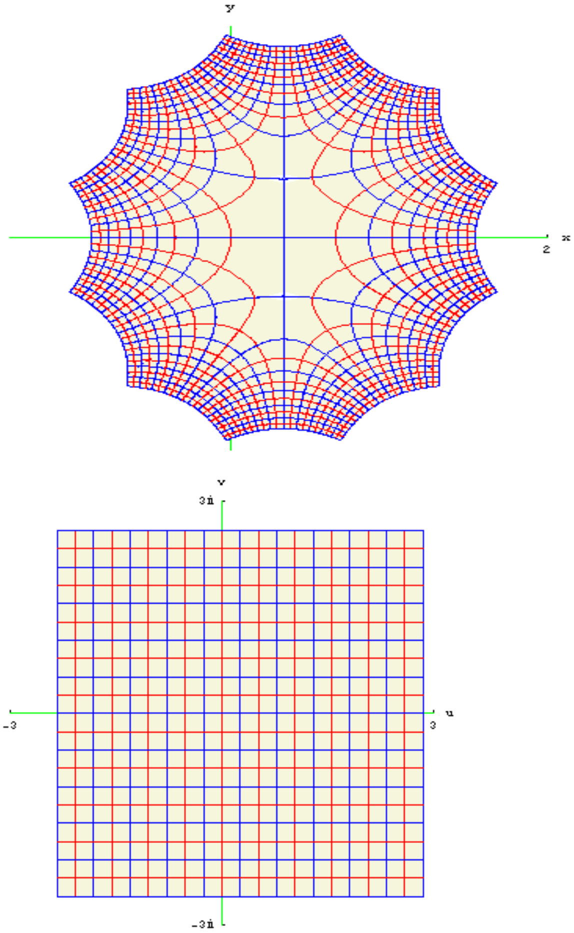
Hence it is applicable when the analytic function is a power series centered about the origin. The reader is encouraged to investigate the origins and limitations of the Milne-Thomson method.

Extra Example 1. Use Methods I and II to construct the harmonic conjugate of $u\,(x,y) = x^3 - 3\,x\,y^2 - x^2 + y^2 + 2$. Also, it shows that the underlying analytic function is $w = f\,(z) = z^3 - z^2 + 2$.



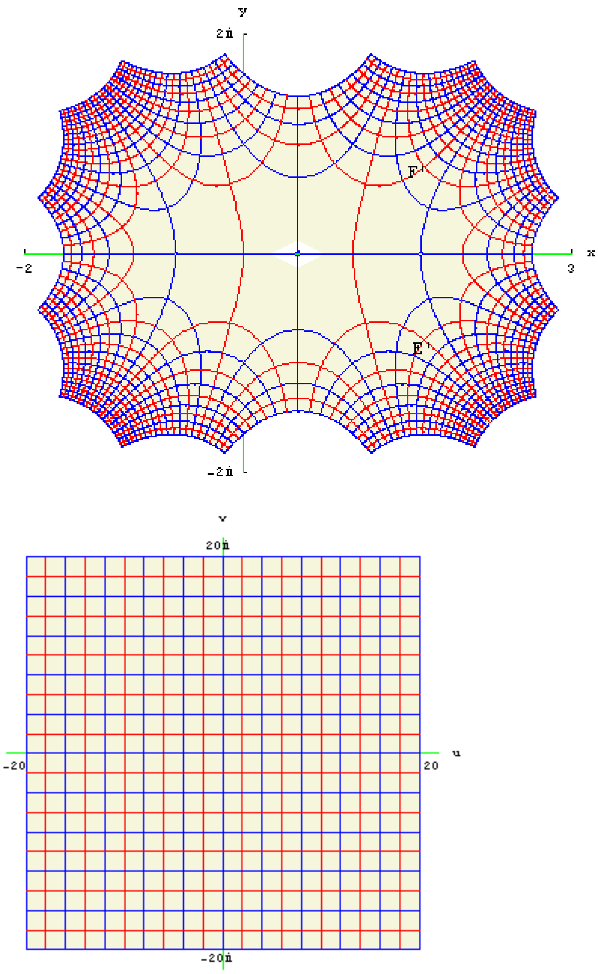
The orthogonal grid in the z-plane and its image under the analytic function $w = f\,(z) = z^3 - z^2 + 2$.

Extra Example 2. Use Methods I and II to construct the harmonic conjugate of $u(x, y) = x^3 - 3xy^2 - x^2 + y^2 + x$. Also, it show that the underlying analytic function is $w = f(z) = z^3 - z^2 + z$.



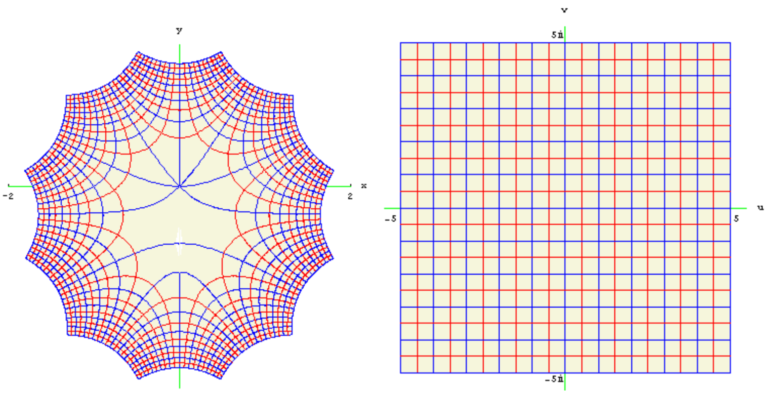
The orthogonal grid in the z-plane and its image under the analytic function $w = f(z) = z^3 - z^2 + z$.

Extra Example 3. Use Methods I and II to construct the harmonic conjugate of $u(x, y) = x^4 + y^4 - 6x^2y^2 - 2x^3 + 6xy^2 - x^2 + y^2 + 2x + 10$. Also, it show that the underlying analytic function is $w = f(z) = z^4 - 2z^3 - z^2 + 2z + 10$.



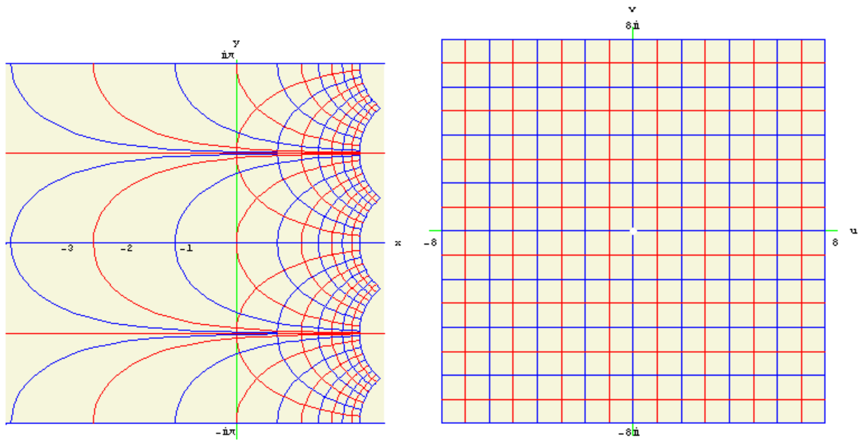
The orthogonal grid in the z-plane and it's image under the analytic function $w = f(z) = z^4 - 2z^3 - z^2 + 2z + 10$.

Extra Example 4. Use Methods I and II to construct the harmonic conjugate of $u(x, y) = x^3 - 3xy^2 - 2xy$. Also, it show that the underlying analytic function is $w = f(z) = z^3 + iz^2$.



The orthogonal grid in the z-plane and it's image under the analytic function $w = f(z) = z^3 + iz^2$.

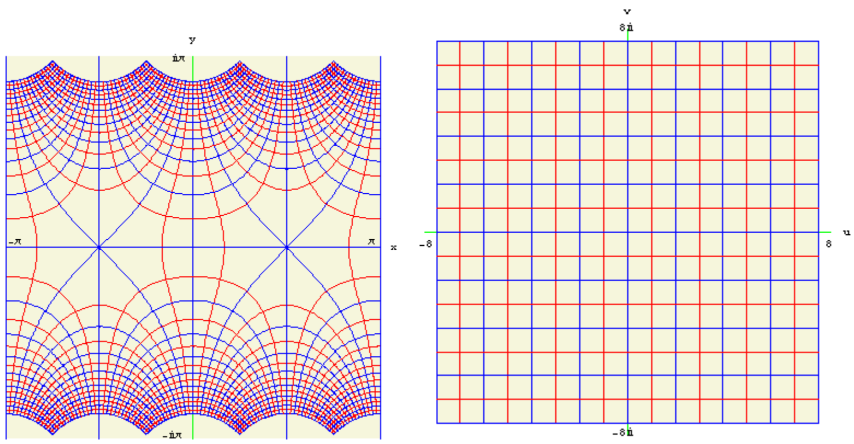
Extra Example 5. Use Methods I and II to construct the harmonic conjugate of $u(x, y) = e^x \cos(y)$. Also, show that the underlying analytic function is $w = f(z) = e^z$.



The orthogonal grid in the z-plane and it's image under the analytic function $w = f(z) = e^z$.

Remark. There are infinitely many branches of the multi-valued inverse function $z = f^{-1}(w)$, and when the regions are combined, they will fill up the z-plane.

Extra Example 6. Use Methods I and II to construct the harmonic conjugate of $u(x, y) = \sin(x)\cosh(y)$. Also, show that the underlying analytic function is $w = f(z) = \sin(z)$.



The orthogonal grid in the z-plane and it's image under the analytic function $w = f(z) = \sin(z)$.

Remark. There are infinitely many branches of the multi-valued inverse function $z = f^{-1}(w)$,

and when the regions are combined, they will fill up the z-plane.

1.4.1 The Complex Plane

The set C of complex numbers is naturally identified with the plane R 2. This is often called the Argand plane. Given a complex number z = x+i y, its real and imag- ✻ ✻ z = x + iy y x ✻ binary parts define an element (x, y) of R 2 , as shown in the figure. In fact, this identification is one of the real vector spaces, in the sense that adding complex numbers and multiplying them with real scalars mimic the similar operations one can do in R 2. Indeed, if α ∈ R is real, then to α z = (α x) + i(α y) there corresponds the pair (α x, α y) = α (x, y). Similarly, if z1 = x1 + i y1 and z2 = x2 + i y2 are complex numbers, then z1 + z2 = (x1 + x2) + i(y1 + y2), whose associated pair is (x1 + x2, y1 + y2) = (x1, y1) + (x2, y2). In fact, the identification is even one of the Euclidean spaces. Given a complex number z = x + i y, its modulus |z|, defined by |z| 2 = zz∗ , is given by p x 2 + y 2 which is precisely the norm k(x, y)k of the pair (x, y). Similarly, if z1 = x1 + i y1 and z2 = x2 + i y2, then Re(z ∗ 1 z2) = x1x2 + y1y2 which is the dot product of the pairs (x1, y1) and (x2, y2). In particular, it follows from these remarks and the triangle inequality for the norm in R 2 , that complex numbers obey a version of the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

1.5 POLAR FORM AND THE ARGUMENT FUNCTION

Points in the plane can also be represented using polar coordinates, and this representation, in turn, translates into a representation of the complex numbers.

Let (x, y) be a point in the plane. If we define r= √(x 2 + y 2) and θ by θ = arctan(y/x), then we can write (x, y) = (r cos θ, r sin θ) = r (cos θ,sin θ). The complex number z = x + i y can then be written as z = r (cos θ + i sin θ). The real number r, as we have seen, is the modulus |z| of z, and the complex number cos θ + i sin θ has unit modulus. Comparing the Taylor series for the cosine and sine functions and the exponential functions we notice that cos θ+isin θ = e iθ. The angle θ is called the argument of z and is written arg(z). Therefore we have the following polar form for a complex number z.

Being an angle, the argument of a complex number is only defined up to the addition of integer multiples of 2π. In other words, it is a multiple-valued function. This ambiguity can be resolved by defining the principal value Arg of the arg function to take values in the interval (−π, π]; that is, for any complex number z, one has

$$-\pi < \mathrm{Arg}(z) \leq \pi.$$

Notice, however, that Arg is not a continuous function: it has a discontinuity along the negative real axis. Approaching a point on the negative real axis from the upper half-plane, the principal value of its argument approaches π, whereas if we approach it from the lower half-plane, the principal value of its argument approaches −π. Notice finally that whereas the modulus is a multiplicative function: |zw| = |z||w|, the argument is additive: arg(z1 z2) = arg(z1) + arg(z2), provided that we understand the equation to hold up to integer multiples of 2π. Also notice that whereas the modulus is invariant under conjugation |z ∗ | = |z|, the argument changes sign arg(z ∗) = − arg(z), again up to integer multiples of 2π.

Check in Progress-II

Note : Please give a solution of questions in space give below:

Q. 1 Define the Complex Plane.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q. 2 Define Argument Function.

Solution :

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…………………………………………………………………………

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1.6 COMPLEX-VALUED FUNCTIONS

In this section, we will discuss complex-valued functions. We start with a rather trivial case of a complex-valued function. Suppose that f is a complex-valued function of a real variable. That means that if x is a real number, f(x) is a complex number, which can be decomposed into its real and imaginary parts: $f(x) = u(x)+i\,v(x)$, where u and v are real-valued functions of a real variable; that is, the objects you are familiar with from calculus. We say that f is continuous at $x0$ if u and v are continuous at $x0$.

Now consider a complex-valued function f of a complex variable z. We say that f is continuous at $z0$ if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z0)| < \varepsilon$ whenever $|z - z0| < \delta$. Heuristically, another way of saying that f is continuous at $z0$ is that f(z) tends to f(z0) as z approaches z0. This is equivalent to the continuity of the real and imaginary parts of f thought of as real-valued functions on the complex plane. Explicitly, if we write $f = u+i\,v$ and $z = x+i\,y$, u(x, y) and v(x, y) are real-valued functions on the complex plane. Then the continuity of f at $z0 = x0 +i\,y0$ is equivalent to the continuity of u and v at the point (x0, y0).

“Graphing” complex-valued functions

Complex-valued functions of a complex variable are harder to visualize than their real analogs. To visualize a real function $f : R \to R$, one simply graphs the function: its graph being the curve $y = f(x)$ in the (x, y)-plane. A complex-valued function of a complex variable $f : C \to C$ maps complex numbers to complex numbers, or equivalently points in the (x, y)-plane to points in the (u, v) plane. Hence its graph defines a surface $u = u(x, y)$ and $v = v(x, y)$ in the four-dimensional space with coordinates (x, y, u, v), which is not so easy to visualize. Instead one resorts to investigating what the function does to regions in the complex plane. Traditionally one considers two planes: the z-plane whose points have coordinates (x, y) corresponding to the real and imaginary parts of z = x + i y, and the w-plane whose points have coordinates (u, v) corresponding to w = u + i v. Any complex-valued function f of the complex variable z maps points in the z-plane to points in the w-plane via w = f(z). A lot can be learned from a complex function by analyzing the image in the w-plane of certain sets in the z-plane. We will have plenty of opportunities to use this throughout the course of these lectures

Differentiability and analyticity

Let us now discuss the differentiation of complex-valued functions. Again, if f = u + i v is a complex-valued function of a real variable x, then the derivative of f at the point x0 is defined by

f 0 (x0) = u 0 (x0) + i v0 (x0) ,

where u 0 and v 0 are the derivatives of u and v respectively. In other words, we extend the operation of differentiation complex-linearly. There is nothing novel here.

Differentiability and the Cauchy–Riemann Equations

The situation is drastically different when we consider a complex-valued function f = u+i v of a complex variable z = x+i y. As is calculus, let us attempt to define its derivative.The first thing that we notice is that Δz, being a complex number, can approach zero in more than one way. If we write Δz = Δx + i Δy, then we can approach zero along with the real axis Δy = 0 or along the imaginary axis Δx = 0, or indeed along any direction. For the derivative to exist, the answer should not depend on how Δz tends to 0. Let us see what this entails. Let us write f = u + i v and z0 = x0 + i y0

we showed that computing the derivative of complex functions written in a form such as $f(z) = z^2$ is a rather simple task. But life isn't always so easy. Many times we encounter complex functions written as (3-13) $f(z) = f(x + i y) = u(x, y) + i v(x, y)$. For example, suppose we had (3-13) $f(z) = f(x + i y) = (x^3 - 3 x y^2) + i (3 x^2 y - y^3)$. Is there some criterion - perhaps involving the partial derivatives of $u(x, y) = x^3 - 3 x y^2$ and $v(x, y) = 3 x^2 y - y^3$, that we can use to

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determine whether $f(z)$ is differentiable and if so, to find the value of $f(z)$?

The answer to this question is yes, thanks to the independent discovery of two important equations by the French mathematician Augustin Louis Cauchy (1789-1857) and the German mathematician Georg Friedrich Bernhard Riemann (1826-1866).

First, let's reconsider the derivative of $f(z) = z^2$. As we have stated, the limit is given in Equation (3-1) must not depend on how z approaches z_0, and a calculation similar to Example 3.1 (in Section 3.1), will prove that $f'(z) = 2z$.

$$f'(z_0) = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(z + z_0)(z - z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} (z + z_0)$$

$$= (z_0 + z_0)$$

$$= 2z_0$$

We can drop the subscript on z_0 to obtain $f'(z) = 2z$ as a general formula.

1.6.1 The Special Cartesian Limits

For the Cartesian coordinate form of a complex function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, it is important to determine how the function values change as we move along the horizontal grid line $z_0 + \Delta z = (x_0 + \Delta x) + iy_0$ at the point $z_0 = x_0 + iy_0$, and how the function values change as we move along the vertical grid line $z_0 + \Delta z = x_0 + i(y_0 + \Delta y)$ at the point $z_0 = x_0 + iy_0$.

We investigate these two approaches: a horizontal approach and a vertical approach to z_0. Recall from our graphical analysis of $w = f(z) = z^2$ in Example 2.12, in Section 2.2, that the image of a square is a "curvilinear quadrilateral" and the images of the horizontal and vertical edges are portions of parabolas in the w-plane. For convenience, we let the square have vertices $z_0 = x_0 + i\,y_0 = 2 + i$, $z_1 = 2.01 + i$, $z_2 = 2 + 1.01\,i$, and $z_3 = 2.01 + 1.01\,i$. Then the image points are $w_0 = u_0 + i\,v_0 = 3 + 4\,i$, $w_1 = 3.0401 + 4.02\,i$, $w_2 = 2.9799 + 4.04\,i$, and $w_3 = 3.02 + 4.0602\,i$, as shown in Figure 3.1.

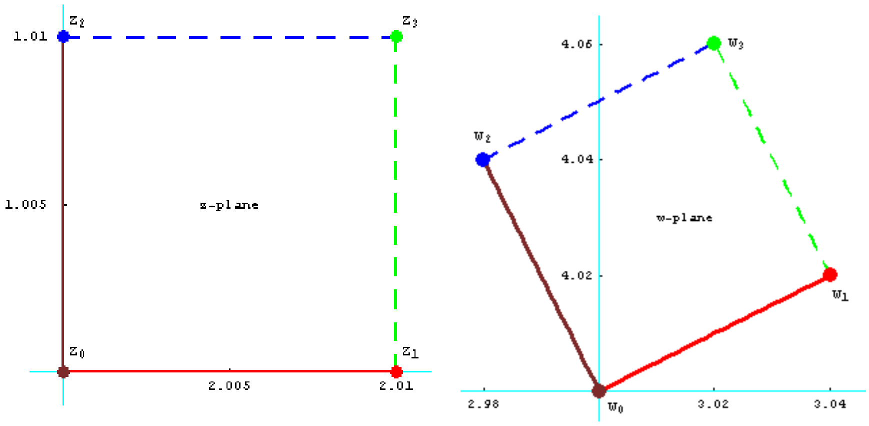

Figure 3.1 The image of a small square under the mapping $w = f(z) = z^2$, the vertex $z_0 = 2 + i$, is mapped onto the point $w_0 = 3 + 4\,i$.

We know that $f(z) = z^2$ is differentiable, so the limit of the difference quotient $\dfrac{f(z) - f(z_0)}{z - z_0}$ exists no matter how we approach $z_0 = x_0 + i\,y_0 = 2 + i$. Let us investigate the two special Cartesian limits.

First, we can numerically approximate $f'(z_0) = f'(2 + i)$ by using a horizontal increment in z.

Use $z_0 = x_0 + i\,y_0 = 2 + i$ and $z_1 = x_0 + \Delta x + i\,y_0 = 2.01 + i$ where

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$\Delta z = \Delta x = 0.01$ to compute the difference quotient.

$$f'(2+i) \approx \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{f(z_1) - f(z_0)}{z_1 - z_0}$$

$$= \frac{f(2.01+i) - f(2+i)}{(2.01+i) - (2+i)}$$

$$= \frac{(3.0401+4.02\,i) - (3+4\,i)}{(2.01+i) - (2+i)}$$

$$= \frac{(3.0401-3) + i\,(4.02-4)}{(2.01-2) - i\,(1-1)}$$

$$= \frac{0.0401 + 0.02\,i}{0.01}$$

$$= 4.01 + 2\,i$$

Second, we can numerically approximate $f'(z_0) = f'(2+i)$ by using a vertical increment in z.

Use $z_0 = x_0 + i\,y_0 = 2+i$ and $z_2 = x_0 + i\,(y_0+\Delta y) = 2+1.01\,i$ where $\Delta z = i\,\Delta y = 0.01\,i$ to compute the difference quotient.

$$f'(2+i) \approx \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{f(z_2) - f(z_0)}{z_2 - z_0}$$

$$= \frac{f(2+1.01\,i) - f(2+i)}{(2+1.01\,i) - (2+i)}$$

$$= \frac{(2.9799+4.04\,i) - (3+4\,i)}{(2+1.01\,i) - (2+i)}$$

$$= \frac{(2.9799-3) + i\,(4.04-4)}{(2-2) + i\,(1.01-1)}$$

$$= \frac{-0.0201 + 0.04\,i}{0.01\,i}$$

$$= 4 + 2.01\,i$$

Comparing these two numerical approximations we see that $f'(2+i) \approx \dfrac{f(z_1) - f(z_0)}{z_1 - z_0} = 4.01 + 2i$, and $f'(2+i) \approx \dfrac{f(z_2) - f(z_0)}{z_2 - z_0} = 4 + 2.01i$, which leads us to speculate that $f'(2+i) = 4 + 2i$.

These numerical approximations lead to the idea of taking limits along the horizontal and vertical directions.

First, we can take the limit along the horizontal direction.

$$\lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(2 + \Delta x + i) - f(2+i)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2 + \Delta x + i)^2 - (3 + 4i)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3 + 4\Delta x + \Delta x^2 + i(4 + 2\Delta x) - (3 + 4i)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\Delta x + \Delta x^2 + 2\Delta x\, i}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (4 + \Delta x + 2i)$$

$$= 4 + 2i$$

Second, we can take the limit along the vertical direction.

$$\lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} = \lim_{\Delta y \to 0} \frac{f(2 + i + i\Delta y) - f(2+i)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(2 + i + i\Delta y)^2 - (3 + 4i)}{i\Delta}$$

$$= \lim_{\Delta y \to 0} \frac{3 - 2\Delta y - \Delta y^2 + (4 + 4\Delta y)\, i - (3 + 4i)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-2\Delta y - \Delta y^2 + 4\Delta y\, i}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} (4 + (2 + \Delta y)\, i)$$

$$= 4 + 2i$$

Notes

Comparing these two limits we see

that $\lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} = 4 + 2 i$, and

$\lim_{\Delta y \to 0} \frac{f(z_0 + i \Delta y) - f(z_0)}{i \Delta y} = 4 + 2 i$.

Since the above two limits were not taken along all possible approaches to $z_0 = 2 + i$, they alone are not sufficient to prove that $f'(2 + i) = 4 + 2 i$, but they prepare our thinking for Theorem 3.3.

Exploration

We now generalize this idea by taking limits of an arbitrary differentiable complex function and obtain an important result.

Theorem 3.3 (Cauchy-Riemann Equations). Suppose that (3-14) $f(z) = f(x + i y) = u(x, y) + i v(x, y)$, is differentiable at the point $z_0 = x_0 + i y_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0), and can be used to calculate the derivative at (x_0, y_0). That is, (3-14) $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$, and also (3-15) $f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$. Equating the real and imaginary parts of Equations (3-14) and (3-15) gives the so-called Cauchy-Riemann Equations: (3-16) $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Exploration for the Cauchy-Riemann Equations.

Aside. Both *Mathematica*™ and Maple™ can assist us in calculating limits.

Aside. The *Mathematica* solution uses the command.

$$\text{Limit}\left[\frac{f[Z] - f[z_0]}{Z - z_0}, Z \to z_0, \text{Analytic} \to \text{True}\right]$$

$f'[z_0]$

$$f[x\_ + i\, y\_] := u[x, y] + i\, v[x, y]$$

$$\text{Limit1} = \text{Limit}\left[\frac{f[x + \Delta x + i\, y] - f[x + i\, y]}{\Delta x}, \Delta x \to 0, \text{Analytic} \to \text{True}\right]$$

$u^{(1,0)}[x, y] + i\, v^{(1,0)}[x, y]$

$$\text{Limit2} = \text{Limit}\left[\frac{f[x + i (y + \Delta y)] - f[x + i y]}{i \Delta y}, \Delta y \to 0, \text{Analytic} \to \text{True}\right]$$

$$-i\, u^{(0,1)}[x, y] + v^{(0,1)}[x, y]$$

Looking at the above limits, and equating the real and imaginary parts we have the following equations. $u^{(1,0)}[x, y] = v^{(0,1)}[x, y]$, and $v^{(1,0)}[x, y] = -u^{(0,1)}[x, y]$.

In *Mathematica* the syntax for partial derivatives can be explained as follows. In the expression $u^{(1,0)}[x, y]$, the superscript $^{(1,0)}$ means, take one derivative with respect to the first variable x. In the expression $u^{(0,1)}[x, y]$, the superscript $^{(0,1)}$ means, take one derivative with respect to the second variable y. Similarly, the expressions $v^{(1,0)}[x, y]$ and $v^{(0,1)}[x, y]$ are the x and y partial derivatives, respectively.

Therefore, we see that *Mathematica* can establish the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

We are done.

Aside. The Maple commands are similar.

$> f := (x, y) \to u(x, y) + I\, v(x, y)$

$f := (x, y) \to u(x, y) + I\, v(x, y)$

$> \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ $\quad D_1(u)(x, y) + I\, D_1(v)(x, y)$

$> \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{I\, \Delta y}$ $\quad -I\, D_2(u)(x, y) + D_2(v)(x, y)$

Looking at the above limits, and equating the real and imaginary parts we have the following equations. $D_1(u)(x, y) = D_2(v)(x, y)$, and $D_1(v)(x, y) = -D_2(u)(x, y)$.

In Maple the syntax for partial derivatives can be explained as follows. In the expression $D_1(u)(x, y)$, the subscript 1 means, take one derivative with respect to the first variable x. In the expression $D_2(u)(x, y)$, the subscript 2 means, take one derivative with respect to the second variable y. Similarly, the expressions $D_1(v)(x, y)$ and $D_2(v)(x, y)$ are the x and y partial derivatives, respectively.

Therefore, we see that Maple can establish the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

1.7 SUMMARY

We study in this unit complex valued function. We study Harmonic Function. We study Harmonic Conjugate function. We study Cauchy Riemann Equation with its examples. We study the limitation of Milne Thomson Method.

1.8 KEYWORD

Integral : A function of which a given function is the derivative, i.e. which yields that function when differentiated, and which may express the area under the curve of a graph of the function

Vortex : A whirling mass of fluid or air, especially a whirlpool or whirlwind

Conjugate : Give the different forms of (a verb in an inflected language such as Latin) as they vary according to voice, mood, tense, number, and person

1.9 QUESTIONS FOR REVIEW

Q. 1 Show that the function w = f(z) = z = x - iy is nowhere differentiable.

Q. 2 If/(z) = z 3 , show we can use definition (1) to get/'(z) = 3z2.

Q. 3 If f is differentiable at zo, then f is continuous at z0.

Q. 4 Let f(z) = fix + iy) = u(x, v) + /V(JC, v) be dijferentiable at the point zo = xo + O^o- Then the partial derivatives of u and v exist at the point (x0, yo) and satisfy the equations.

ux(x0, y0) = VvUo, yo) and i/v(x0, y0) = -vv(x0, y0).

Q. 5 The function/(z) = z 3 = x 3 - 3xy2 + /(3xy2 - y3) is known to be differentiable.

Q. 6 Show that the following functions are entire. (a) f(z) = cosh x sin y – i sinh x cos y

(b) g{z) = cosh x cos y + i sinh x sin y

Q. 7 Let u(x, y) be harmonic. Show that U(x, y) = u(x, —y) is harmonic. Hint: Use the chain rule for differentiation of real functions.

1.10 NOTES

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1.11 SUGGESTION READING AND REFERENCES

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1.12 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1

2 Check in Section 3

Check In Progress-II

Answer Q. 1 check in section 5.1

Answer Q 2 Check in Section 6

UNIT 2: ANALYTIC & HARMONIC FUNCTION

STRUCTURE

2.0 OBJECTIVE

Does the notion of a derivative of a complex function make sense? If so, how should it be defined and what does it represent? These and similar questions are the focus of this chapter. As you might guess, complex derivatives have a meaningful definition, and many of the standard derivative theorems from calculus (such as the product rule and chain rule) carry over for complex functions. There are also some interesting applications. But not everything is symmetric. You will learn in this chapter that the mean value theorem or derivatives do not extend to complex functions. In later chapters, you will see that differentiable complex functions are, in some sense, much more "differentiable" than differentiable real functions.

2.1 INTRODUCTION

I think that a real function u(x,y) is harmonic if it obeys that equation. If it does, then there is another real function v(x,y) that is also harmonic, and there is a complex function f(x+iy)=u(x,y)+iv(x,y) which is differentiable. By that, I mean, you can write $f(x+iy)=g(x+iy,x-iy)=g(z,\overline{z})$

and $\partial g/\partial \overline{z}=0$

The harmonic functions are those satisfying the Laplace equation Δu=0, where Δ≡∂2x+∂2y is the Laplace operator. Usually one assumes them to be of class C2 (defined on some open subset of the complex plane, say, and taking real values; of course one can consider more general situations), but since any harmonic function admits (locally, which is enough of course) a harmonic conjugate, they are automatically of class C∞.

2.1.1 Analytic Functions

Using our imagination, we take our lead from elementary calculus and define the derivative of $f(z)$ at z_0, written $f'(z_0)$, by (3-1) $f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, provided that the limit exists. If it does, we say that the function $f(z)$ is differentiable at z_0. If we write $\Delta z = z - z_0$, then we can express Equation (3-1) in the form (3-2) $f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$. If we let $w = f(z)$ and $\Delta w = f(z) - f(z_0)$, then we can use the Leibniz's notation $\frac{dw}{dz}$ for the derivative: (3-3) $f'(z_0) = \frac{dw}{dz} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z}$.

Example 3.1. Use the limit definition to find the derivative of $f(z) = z^3$.

Solution. Using Equation (3-1), we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(z^2 + z z_0 + z_0^2)(z - z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} (z^2 + z z_0 + z_0^2)$$

$$= (z_0^2 + z_0 z_0 + z_0^2)$$

$$= 3 z_0^2$$

We can drop the subscript on z_0 to obtain $f'(z) = 3z^2$ as a general formula.

Alternative Solution. Using Equation (3-2), we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - z_0^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3 z_0^2 \Delta z + 3 z_0 \Delta z^2 + \Delta z^3 - z_0^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3 z_0^2 \Delta z + 3 z_0 \Delta z^2 + \Delta z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (3 z_0^2 + 3 z_0 \Delta z + \Delta z^2)$$

$$= (3 z_0^2 + 3 z_0 \times 0 + 0^2)$$

$$= 3 z_0^2$$

We can drop the subscript on z_0 to obtain $f'(z) = 3z^2$ as a general formula.

Pay careful attention to the complex value Δz in Equation (3-3); the value of the limit must be independent of the manner in which $\Delta z \to 0$. If we can find two curves that end at z_0 along which $\frac{\Delta w}{\Delta z}$ approaches two

Notes

distinct values, then $\frac{\Delta w}{\Delta z}$ does not have a limit as $\Delta z \to 0$ and $f(z)$ does not have a derivative at z_0. The same observation applies to the limits in Equations (3-2) and (3-1).

Example 3.2. Show that the function $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We choose two approaches to the point $z_0 = x_0 + i y_0$ and compute limits of the two difference quotients. We shall use formulas similar to (3-1), for calculating the directional derivatives along horizontal and vertical lines.

First, we approach $z_0 = x_0 + i y_0$ along a line parallel to the x-axis by forcing z to be of the form $z = x + i y_0$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{f(x + i y_0) - f(x_0 + i y_0)}{(x + i y_0) - (x_0 + i y_0)}$$

$$= \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{(x - i y_0) - (x_0 - i y_0)}{(x + i y_0) - (x_0 + i y_0)}$$

$$= \lim_{x \to x_0} \frac{(x - i y_0) - (x_0 - i y_0)}{(x - x_0) + i (y_0 - y_0)}$$

$$= \lim_{x \to x_0} \frac{x - x_0}{x - x_0}$$

$$= 1$$

.

Next, we approach z_0 along a line parallel to the y-axis by forcing z to be of the form $z = x_0 + i y$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x_0+iy) \to (x_0+iy_0)} \frac{f(x_0 + i y) - f(x_0 + i y_0)}{(x_0 + i y) - (x_0 + i y_0)}$$

$$= \lim_{(x_0+iy) \to (x_0+iy_0)} \frac{(x_0 - i y) - (x_0 - i y_0)}{(x_0 + i y) - (x_0 + i y_0)}$$

$$= \lim_{y \to y_0} \frac{(x_0 - i y) - (x_0 - i y_0)}{(x_0 - x_0) + i (y - y_0)}$$

$$= \lim_{y \to y_0} \frac{-i (y - y_0)}{i (y - y_0)}$$

$$= -1$$

.

The limits along the two paths are different, so there is no possible value for the right side of Equation (3-1). Therefore $f(z) = \overline{z}$ is not differentiable at the point z_0, and since z_0 was arbitrary, $f(z) = \overline{z}$ is nowhere differentiable.

Remark 3.1. In Section 2.3 we showed that $f(z) = \overline{z}$ is continuous for all z. Thus, we have a simple example of a function that is continuous everywhere but differentiable nowhere. Such functions are hard to construct in real variables. In some sense, the complex case has made pathological constructions simpler!

We are seldom interested in studying functions that aren't differentiable, or even differentiable at only a single point. Complex functions that have a derivative at all points in a neighborhood of z_0 deserve further study. In Section 7.2, we will prove that, if the complex function $f(z)$ can be represented by a Taylor series at z_0, then it must be differentiable in some neighborhoods of z_0. Functions that are differentiable in neighborhoods of points are pillars of the complex analysis edifice; we give them a special name, as indicated in the following definition.

Definition 1.1 (Analytic Function). The complex function $f(z)$ is analytic at the point z_0 provided there is some $\epsilon > 0$ such that $f'(z)$ exists for all $z \in D_\epsilon(z_0)$. In other words, $f(z)$ must be differentiable not only at z_0, but also at all points in some ϵ-neighborhood of z_0.

If $f(z)$ is analytic at each point in the region R, then we say that $f(z)$ is an analytic function on R. Again, we have a special term if $f(z)$ is analytic on the whole complex plane.

2.2 ENTIRE FUNCTION

Definition 1.2 (Entire Function). If $f(z)$ is analytic on the whole complex plane then $f(z)$ is said to be an entire function.

Notes

Points of non-analyticity for a function are called singular points. They are important for applications in physics and engineering.

Our definition of the derivative for complex functions is formally the same as for real functions and is the natural extension from real variables to complex variables. The basic differentiation formulas are identical to those for real functions, and we obtain the same rules for differentiating powers, sums, products, quotients, and compositions of functions. We can easily establish proof of the differentiation formulas by using the limit theorems.

2.2.1 The Rules for Differentiation.

Suppose that f(z) and g(z) are differentiable. From Equation (3-2) and the technique exhibited in the solution to Example 3.1, we can establish the following rules, which are virtually identical to those for real-valued functions. (3-4) $\frac{d}{dz}(a + ib) = 0$, where $(a + ib)$ is a constant, (3-5) $\frac{d}{dz} z^n = n z^{n-1}$, where n is a positive integer, (3-6) $\frac{d}{dz}((a + ib) f(z)) = (a + ib) f'(z)$,

(3_7)

$\frac{d}{dz}((a_1 + ib_1) f(z) + (a_2 + ib_2) g(z)) = (a_1 + ib_1) f'(z) + (a_2 + ib_2) g'(z)$, (3-8) $\frac{d}{dz}(f(z) g(z)) = f'(z) g(z) + f(z) g'(z)$, (3-9) $\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$, provided that $g(z) \neq 0$, (3-10) $\frac{d}{dz} f(g(z)) = f'(g(z)) g'(z)$.

Important particular cases of Equations (3-9) and (3-10), respectively, are (3-11) $\frac{d}{dz}\frac{1}{z^n} = \frac{-n}{z^{n+1}}$, for $z \neq 0$, and n is a positive integer, (3_12) $\frac{d}{dz}(f(z))^n = n(f(z))^{n-1} f'(z)$, where n is a positive integer.

Example 1.3. Use Formula (3-12) to calculate $\frac{d}{dz}(z^2 + i2z + 3)^4$.

Hint. Use $f(z) = z^2 + i2z + 3$, $f'(z) = 2z + 2i$, and $n = 4$.

Solution. An easy computation yields

$$\frac{d}{dz}(z^2 + i2z + 3)^4 = \frac{d}{dz}(f(z))^4$$
$$= 4(f(z))^3 f'(z)$$
$$= 4(z^2 + i2z + 3)^3 (2z + 2i)$$
$$= 8(z^2 + i2z + 3)^3 (z + i)$$

The proofs of the rules given in Equations through (3-10) depend on the validity of extending theorems for real functions to their complex companions. Equation (3-8), for example, relies on Theorem 3.1.

Theorem 3.1. If $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0.

Proof. From Equation (3-1), we obtain $\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$. Using the multiplicative property of limits given in Theorem 2.3 in Section 2.3, we

$$\lim_{z\to z_0}(f(z) - f(z_0)) = \lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0}(z - z_0)$$
$$= \lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0}\lim_{z\to z_0}(z - z_0)$$
$$= f'(z_0)\times 0$$
$$= 0$$

get

This result implies that $\lim_{z\to z_0} f(z) - \lim_{z\to z_0} f(z_0) = 0$, which in turn implies that $\lim_{z\to z_0} f(z) = f(z_0)$. Therefore, $f(z)$ is continuous at z_0.

The Derivative of $f(z)g(z)$

We can establish Equation (3-8)

$\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z)$, from Theorem 3.1.

Notes

Letting $h(z) = f(z)g(z)$ and using Definition 3.1, we write

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0}.$$

If we subtract and add the term $f(z_0)g(z)$ in the numerator, we get

$$h'(z_0) = \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z)}{z - z_0} + \lim_{z \to z_0} \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(f(z) - f(z_0))g(z)}{z - z_0} + \lim_{z \to z_0} \frac{f(z_0)(g(z) - f(z_0)g(z_0))}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} g(z) + \lim_{z \to z_0} f(z_0) \lim_{z \to z_0} \frac{g(z) - f(z_0)g(z_0)}{z - z_0}$$

Using the definition of the derivative given by Equation (3-1) and the continuity of $g(z)$, we obtain $h'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$, which is what we wanted to establish.

We leave the proofs of the other differentiation rules as exercises.

The rule for differentiating polynomials carries over to the complex case as well. If we let $P(z)$ be a polynomial of degree n, so that $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n$, then mathematical induction, along with Equations (3-5) and (3-7), gives $P'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + a_{n-1} z^{n-1}$. Again, we leave the details of this proof for the reader to finish, as an exercise.

We shall use the differentiation rules as aids in determining when functions are analytic. For example, Equation (3-9) tells us that if $P(z)$ and $Q(z)$ are polynomials, then their quotient $\frac{P(z)}{Q(z)}$ is analytic at all points where $Q(z) \neq 0$. This condition implies that the function $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$.

The square root function is more complicated. If $f(z) = z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\frac{Arg(z)}{2}}$, then $f(z)$ is analytic at all points except $z = 0$ (because $Arg(0)$ is undefined) and at points that lie along with the negative x-axis. Recall from Exercise 17, in Section 2.3, that the argument function is not continuous along with the negative x-axis. Therefore the function $f(z) = z^{\frac{1}{2}}$, is not continuous at points that lie along with the negative x-axis.

We close this section with a complex extension of a famous theorem, which is attributed to Guillaume de l'Hôpital (1661-1704), the proof will be given in Section 7.5.

Theorem 3.2 (L'Hôpital's Rule). Assume that $f(z)$ and $g(z)$ are both analytic at z_0. If $f(z_0) = 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$, then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$.

Extra Example 1. Use L'Hôpital's rule to find $\lim_{z \to 1+i} \frac{z^2 + z - 1 - 3i}{z^2 - 2z + 2}$.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 Define Analytic Function.

Solution :

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Q. 2 Give Definition of Entire Function.

Solution :

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Notes

2.3 REAL CONCEPTS IN COMPLEX ANALYSIS.

Many of the calculus concepts about derivatives are easy to extended to complex functions. For example, in calculus we learned that the derivative is the limit of the difference quotients $\frac{f(x+\Delta x) - f(x)}{\Delta x}$ as Δx goes to zero. We can compare our calculus experience with some new and interesting graphs in the complex plane.

Extra Example 2. Consider the real function $f(x) = \frac{x^6}{6} + 2x^3$, which is differentiable, and it's derivative is the limit of the real difference quotients $\frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

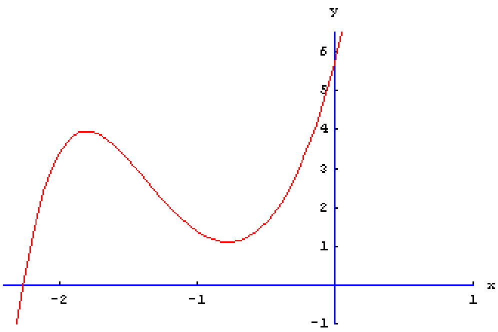
$$= \lim_{\Delta x \to 0} \frac{\frac{1}{6}(x+\Delta x)^6 + 2(x+\Delta x)^3 - \frac{1}{6}x^6 - 2x^3}{\Delta x}$$

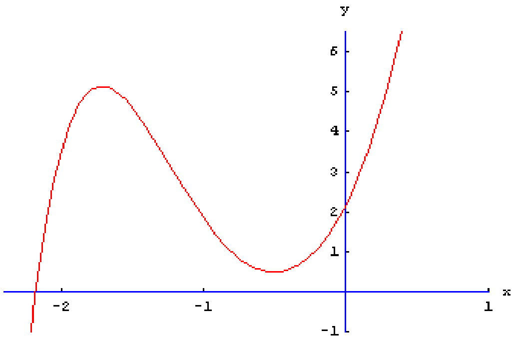
$$= \lim_{\Delta x \to 0} \frac{(x^5 + 6x^2)\Delta x + \left(\frac{5x^4}{2} + 6x\right)\Delta x^2 + \left(\frac{10x^3}{3} + 2\right)\Delta x^3 + \frac{5x^2}{2}\Delta x^4 + x\,\Delta x^5 + \frac{1}{6}\Delta x^6}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(x^5 + 6x^2 + \left(\frac{5x^4}{2} + 6x\right)\Delta x + \left(\frac{10x^3}{3} + 2\right)\Delta x^2 + \frac{5x^2}{2}\Delta x^3 + x\,\Delta x^4 + \frac{1}{6}\Delta x^5\right)$$

$$= x^5 + 6x^2$$

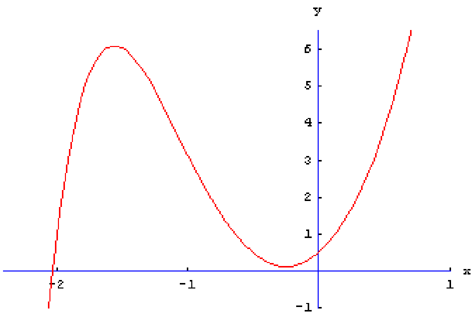
We can illustrate convergence of the real difference quotients $\frac{f(x+\Delta x) - f(x)}{\Delta x}$ by comparing graphs for decreasing values of Δx. For illustration purposes we plot the real graphs $y = \frac{f(x+\Delta x) - f(x)}{\Delta x}$ for $\Delta x = 1.5, 1.0, 0.5, 0.1$.

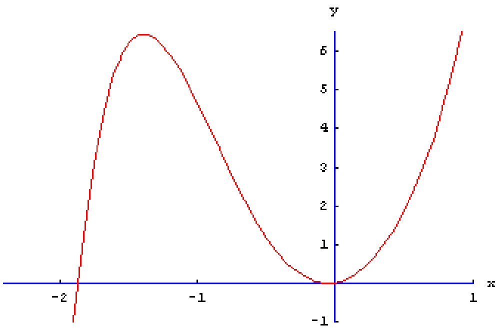




$$y = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ for } \Delta x = \frac{3}{2}$$

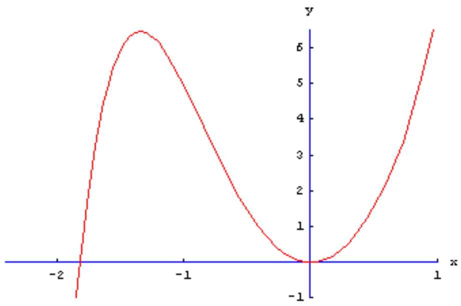
$$y = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ for } \Delta x = 1$$





$$y = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ for } \Delta x = \frac{1}{2}$$

$$y = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ for } \Delta x = \frac{1}{10}$$



The graph of $y = f'(x) = x^5 + 6x^2$.

Figure E.E.3. The graphs

of $y = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ for $\Delta x = \dfrac{3}{2}$, 1, $\dfrac{1}{2}$ and $\dfrac{1}{10}$. where

$f(x) = \dfrac{1}{6}x^6 + 2x^3$ and the graph of $y = f'(x) = x^5 + 6x^2$.

By looking at the above graphs we should get a good feeling about visualizing limits of functions over an interval. In particular, we hope that this gives you a good feeling about the the formula

$f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$, where we have used the function

$f(x) = \dfrac{1}{6}x^6 + 2x^3$ in this illustration to get

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = x^5 + 6x^2.$$

The real function $f(x)$ can be extended into the complex plane by replacing the real variable x with the complex variable x. The same algebraic computations are involved in finding the limit of the complex difference quotients.

Extra Example 3. Consider the complex function $f(z) = \dfrac{z^6}{6} + 2z^3$, which is differentiable, and it's derivative is the limit of the complex difference quotients $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

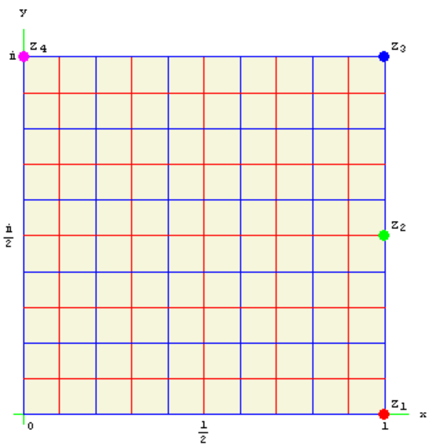
$$= \lim_{\Delta z \to 0} \frac{\frac{1}{6}(z+\Delta z)^6 + 2(z+\Delta z)^3 - \frac{1}{6}z^6 - 2z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z^5+6z^2)\Delta z + \left(\frac{5z^4}{2}+6z\right)\Delta z^2 + \left(\frac{10z^3}{3}+2\right)\Delta z^3 + \frac{5z^2}{2}\Delta z^4 + z\Delta z^5 + \frac{1}{6}\Delta z^6}{\Delta z}$$

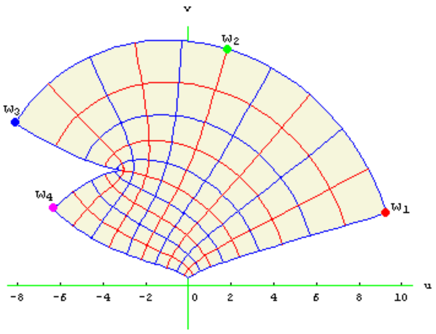
$$= \lim_{\Delta z \to 0} \left(z^5+6z^2+\left(\frac{5z^4}{2}+6z\right)\Delta z + \left(\frac{10z^3}{3}+2\right)\Delta z^2 + \frac{5z^2}{2}\Delta z^3 + z\Delta z^4 + \frac{1}{6}\Delta z^5\right)$$

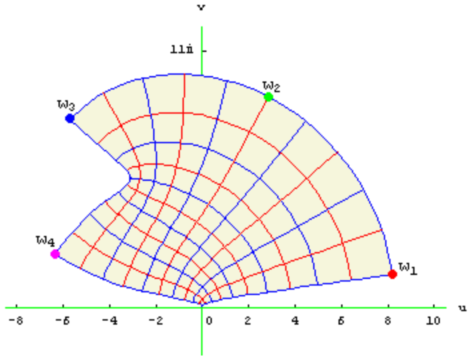
$$= z^5 + 6z^2$$

We can illustrate convergence of the complex difference quotients $\frac{f(z+\Delta z) - f(z)}{\Delta z}$ by comparing graphs for decreasing values of Δz. For illustration purposes we plot the graphs $w = \frac{f(z+\Delta z) - f(z)}{\Delta z}$ for $\Delta z = 0.4\frac{1+i}{\sqrt{2}},\ 0.2\frac{1+i}{\sqrt{2}},\ 0.1\frac{1+i}{\sqrt{2}},\ 0.05\frac{1+i}{\sqrt{2}}$. We cannot draw a graph of 2-dimensional space into 2-dimensional space, it is necessary to choose a domain D in the z-plane for our graphs.



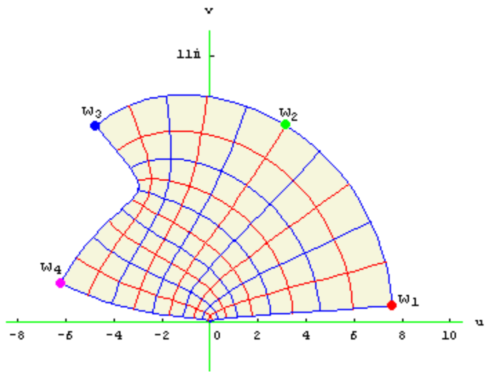
The domain D in the z-plane for the following graphs.

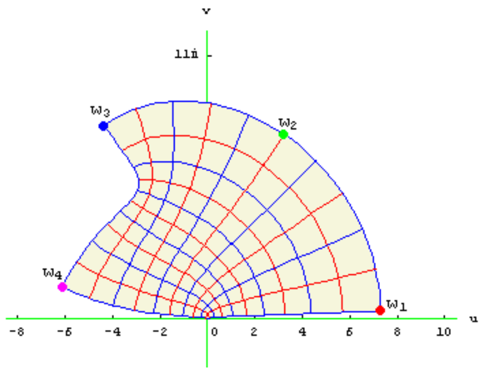




$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ for } \Delta z = 0.40\,\frac{1+i}{\sqrt{2}}$$

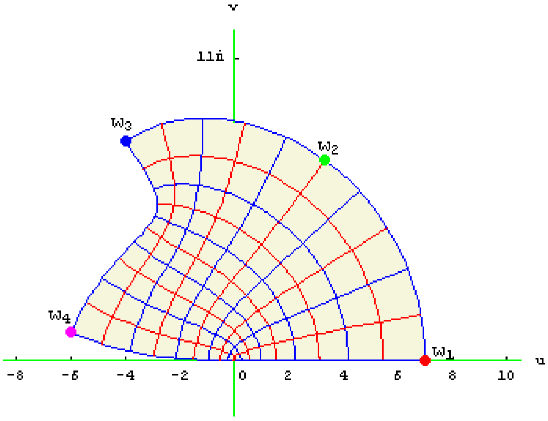
$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ for } \Delta z = 0.20\,\frac{1+i}{\sqrt{2}}$$





$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ for } \Delta z = 0.10\,\frac{1+i}{\sqrt{2}}$$

$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ for } \Delta z = 0.05\,\frac{1+i}{\sqrt{2}}$$



The graph of $w = f'(z) = z^5 + 6z^2$.

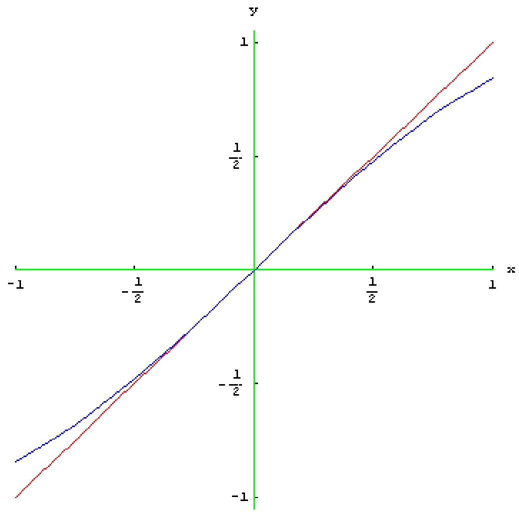
Figure E.E.4. The unit square in the z-plane, and it's images under the mappings $w = \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ for $\Delta z = 0.4\dfrac{1+i}{\sqrt{2}},\ 0.2\dfrac{1+i}{\sqrt{2}},\ 0.1\dfrac{1+i}{\sqrt{2}},\ 0.05\dfrac{1+i}{\sqrt{2}}$, where $f(z) = \dfrac{1}{6}z^6 + 2z^3$ and the graph of $w = f'(z) = z^5 + 6z^2$.

By looking at the above graphs we should get a good feeling about visualizing limits of functions in the complex plane. In particular, we hope that this gives you a good feeling about the the formula $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$, where we have used the function $f(z) = \dfrac{1}{6}z^6 + 2z^3$ in this illustration to get $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = z^5 + 6z^2$.

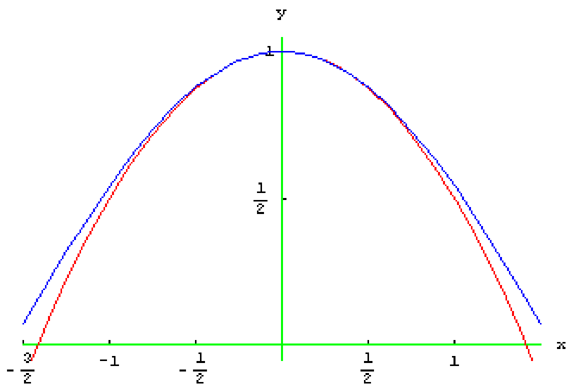
Remark. The final resting place of the points w_1, w_2, w_3, w_4 are $w_1 = f'(1) = 7$, $w_2 = f'\left(1 + \dfrac{i}{2}\right) = \dfrac{53}{16} + \dfrac{233}{32}i$, $w_3 = f'(1+i) = -4 + 8i$, and $w_4 = f'(i) = -6 + i$.

2.4 GRAPHICAL EXPLORATIONS OF POLYNOMIAL APPROXIMATIONS

Many concepts from calculus will be extended to complex functions, including the approximation of functions. Derivatives will play an important role, just as they did in the calculus of real functions. Let us give a preview of some things we will be studying. The following three polynomial approximations are usually discussed in calculus.

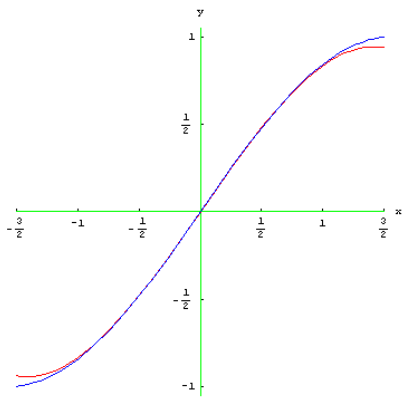


$y = p_1(x) = x$ is an approximation to $y = f(x) = \sin(x)$.



$y = p_2(x) = 1 - \frac{1}{2!}x^2$ is an approximation to $y = f(x) = \cos(x)$

.

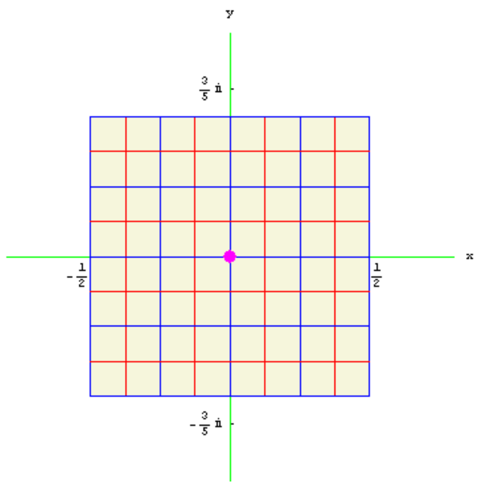


$y = p_2(x) = x - \frac{1}{3!}x^3$ is an approximation to $y = f(x) = \sin(x)$.

In the above graphs, is easy to visualize the real functions and their polynomial approximations $\sin(x) \approx p_1(x) = x$, $\cos(x) \approx p_2(x) = 1 - \frac{1}{2!}x^2$, $\sin(x) \approx p_2(x) = x - \frac{1}{3!}x^3$. We assume that the reader is familiar with the details for constructing these approximations, or can easily find them.

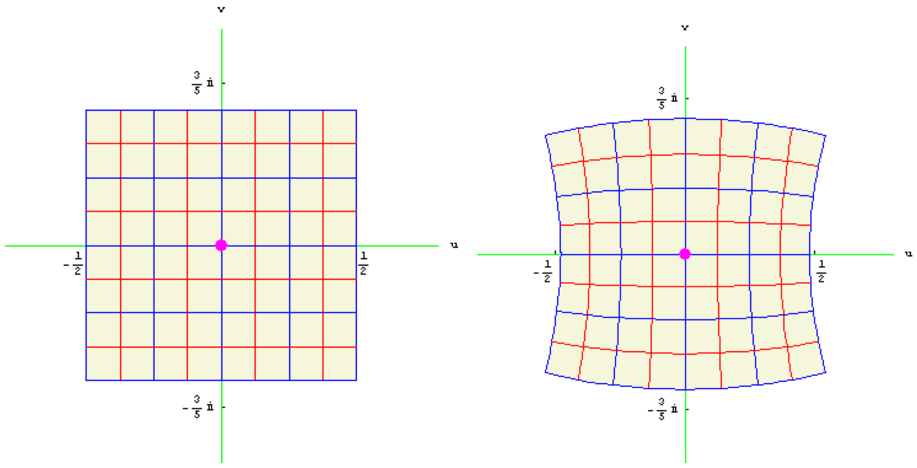
However, when we extend these real functions to complex functions, we must select an appropriate domain D for each function in the z-plane in order to construct it's a graph. The following complex function examples give illustrations similar to the above real approximations but extended into the complex plane.

Extra Example 4. Given $f(z) = \sin(z)$, from calculus we know that $f(0) = 0$, $f'(0) = 1$. The Maclaurin polynomial of degree $n = 1$ is $p_1(z) = f(0) + f'(0)z$. Hence, the mapping $w = f(z) = \sin(z)$, has the "linear approximation" $w = p_1(z) = z$.



The domain D in the z-plane for the following graphs.

Notes



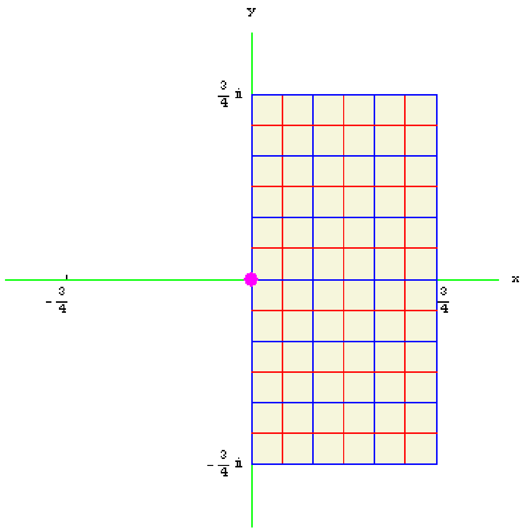
$w = p_1(z) = z$ $\quad$ $w = f(z) = \sin(z)$

Figure E.E.5. The domain $D = \left\{z = x + i y : -\frac{1}{2} < x < \frac{1}{2},\ -\frac{1}{2} < y < \frac{1}{2}\right\}$ is a square in the z-plane, and it images under the mappings $w = s_1(z) = z$ and $w = f(z) = \sin(z)$.

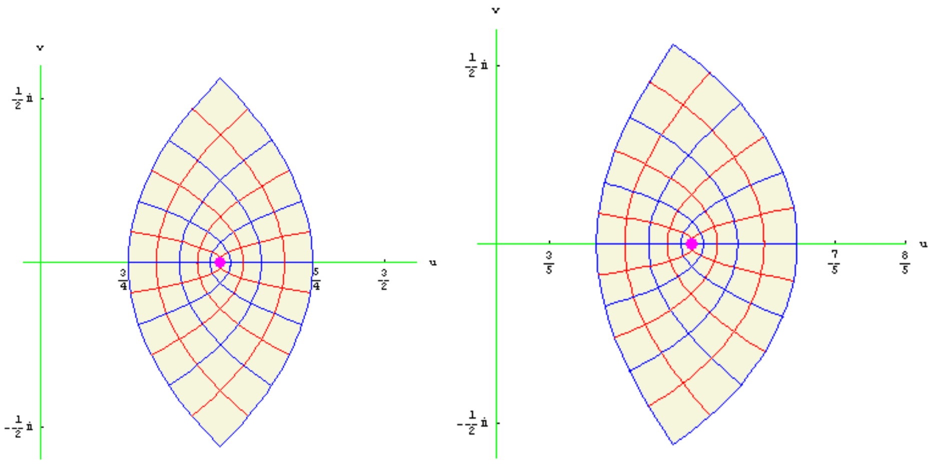
In the last two graphs, one can visualize the complex function approximation $\sin(z) \approx p_1(z) = z$.

Remark 1. This is a trivial example of a "linear transformation" that was studied in Section 2.1. Also, $p_1(z) = z$ is a "linear approximation" to $w = f(z) = \sin(z)$. **Remark 2.** Complex Taylor polynomials and approximations will be introduced in Section 7.2. The function $p_1(z) = z$ is the familiar Maclaurin polynomial approximation of degree $n = 1$. **Remark 3.** Analytic functions that satisfy $f'(z) \neq 0$ are conformal mappings and will be studied in Section 10.1.

Extra Example 5. Given $f(z) = \cos(z)$, from calculus we know that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$. The Maclaurin polynomial of degree $n = 2$ is $p_2(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2$. Hence, the mapping $w = f(z) = \cos(z)$, has the "quadratic approximation" $w = p_2(z) = 1 - \frac{1}{2}z^2$.



The domain D in the z-plane for the following graphs.



$w = p_2(z) = 1 - \frac{1}{2!} z^2$ $\qquad w = f(z) = \cos(z)$

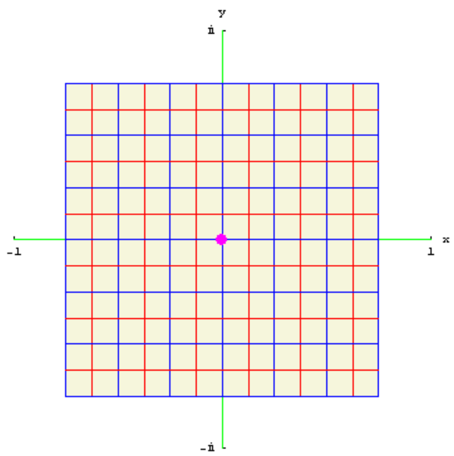
Figure E.E.6. The domain $D = \left\{z = x + iy : 0 < x < \frac{3}{4},\ -\frac{3}{4} < y < \frac{3}{4}\right\}$ is a rectangle in the z-plane, and it's images under the mappings $w = p_2(z) = 1 - \frac{1}{2!} z^2$ and $w = f(z) = \cos(z)$.

In the last two graphs, one can visualize the complex function approximation $\cos(z) \approx p_2(z) = 1 - \frac{1}{2!} z^2$.

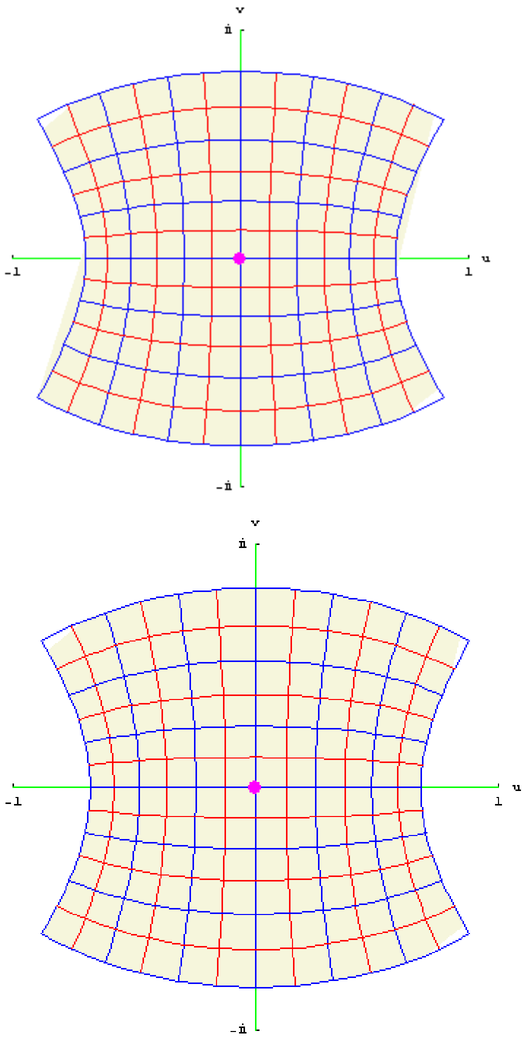
Notes

Remark 4. The mapping $w = p_2(z) = 1 - \frac{1}{2!}z^2$ is similar to the mapping $w = z^2$ that was studied in Section 2.2. Also, $p_2(z) = 1 - \frac{1}{2!}z^2$ is a "quadratic approximation" to $w = f(z) = \cos(z)$. **Remark 5.** Complex Taylor polynomials and approximations will be introduced in Section 7.2. The function $p_2(z) = 1 - \frac{1}{2!}z^2$ is the familiar Maclaurin polynomial approximation of degree $n = 2$. **Remark 6.** Analytic functions that satisfy $f'(z) \neq 0$ are conformal mappings and will be studied in Section 10.1. We will see that the mapping $w = z^2$ is not conformal at the origin.

Extra Example 6. Given $f(z) = \sin(z)$, from calculus we know that the first few derivatives are $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$. The Maclaurin polynomial of degree $n = 3$ is $p_3(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^2$. Hence, the mapping $w = f(z) = \sin(z)$, has the "cubic approximation" $w = p_3(z) = z - \frac{1}{3!}z^3$.



The domain D in the z-plane for the following graphs.



$w = p_3(z) = z - \frac{1}{3!} z^3$ $\qquad$ $w = f(z) = \sin(z)$

Figure E.E.7. The domain $D = \left\{ z = x + i y : -\frac{3}{4} < x < \frac{3}{4},\ -\frac{3}{4} < y < \frac{3}{4} \right\}$ is a square in the z-plane, and it's images under the mappings $w = p_3(z) = z - \frac{1}{3!} z^3$ and $w = f(z) = \sin(z)$.

In the last two graphs, one can visualize the complex function approximation $\sin(z) \approx p_3(z) = z - \frac{1}{3!} z^3$.

Remark 7. Complex Taylor polynomials and approximations will be introduced in Section 7.2. The function $p_3(z) = z - \frac{1}{3!} z^3$ is the familiar Maclaurin polynomial approximation of degree $n = 3$.

Remark 8. Analytic functions that satisfy $f'(z) \neq 0$ are conformal mappings and will be studied in below :

The terminology "conformal mapping" should have a familiar sound. In 1569 the Flemish cartographer Gerardus Mercator (1512--1594) devised a cylindrical map projection that preserves angles. The Mercator projection is still used today for world maps. Another map projection known to the ancient Greeks is the stereographic projection. It is also conformal (i.e., angle preserving), and we introduced it in Section 2.5 when we defined the Riemann sphere. In complex analysis, a function preserves angles if and only if it is analytic or anti-analytic (i.e., the conjugate of an analytic function). A significant result, known as Riemann mapping theorem, states that any simply connected domain (other than the entire complex plane) can be mapped conformally onto the unit disk.

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 Define polynomial Approximation.

Solution :

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……………………………………………………………………………
….……………………………………………………………………………

Q. 2 Define limit of complex analysis.

Solution :

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2.5 BASIC PROPERTIES OF CONFORMAL MAPPINGS

Let f(z) be an analytic function in the domain D, and let z_0 be a point in D. If $f'(z_0) \neq 0$, then we can express f(z) in the form (10-1)

$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$, where $\eta(z) \to 0$ as $z \to z_0$.

If z is near z_0, then the transformation $w = f(z)$ has the linear approximation $S(z) = A + B(z - z_0)$, where $A = f(z_0)$ and $B = f'(z_0)$. Because of $\eta(z) \to 0$ when $z \to z_0$, for points near z_0 the transformation $w = f(z)$ has an effect much like the linear mapping $w = S(z)$. The effect of the linear mapping S is a rotation of the plane through the angle $\alpha = \mathrm{Arg}(f'(z_0))$, followed by a magnification by the factor $|f'(z_0)|$, followed by a rigid translation by the vector $A + Bz_0$. Consequently, the mapping $w = S(z)$ preserves angle at the point z_0. We now show that the mapping $w = f(z)$ also preserves angles at z_0. For a smooth curve that passes through the point $z(0) = z_0$, we use the notation $C: z(t) = x(t) + iy(t)$, for $-1 \leq t \leq 1$. A vector $\vec{T}$ tangent to C at the point z_0 is given by $\vec{T} = z'(0)$, where the complex number $z'(0)$ is expressed as a vector.

The angle of inclination of $\vec{T}$ with respect to the positive x axis is $\beta = \mathrm{Arg}\, z'(0)$.

The image of C under the mapping $w = f(z)$ is the curve K in the w plane given by the formula $K: w(t) = u(x(t), y(t)) + iv(x(t), y(t))$. We can use the chain rule to show that a vector $\vec{T^*}$ tangent to K at the point $w_0 = f(z_0)$ is given by $\vec{T^*} = w'(0) = f'(z_0)z'(0)$.

The angle of inclination of $\vec{T^*}$ with respect to the positive u axis is $\gamma = \mathrm{Arg}\, w'(0) = \mathrm{Arg}\, f'(z_0) + \mathrm{Arg}\, z'(0) = \alpha + \beta$, where $\alpha = \mathrm{Arg}\, f'(z_0)$.

Therefore the effect of the transformation $w = f(z)$ is to rotate the angle of inclination of the tangent vector $\vec{T}$ at z_0 through the angle $\alpha = \mathrm{Arg}\, f'(z_0)$ to obtain the angle of inclination of the tangent vector $\vec{T^*}$ at $w_0 = f(z_0)$. This situation is illustrated in Figure 10.1.

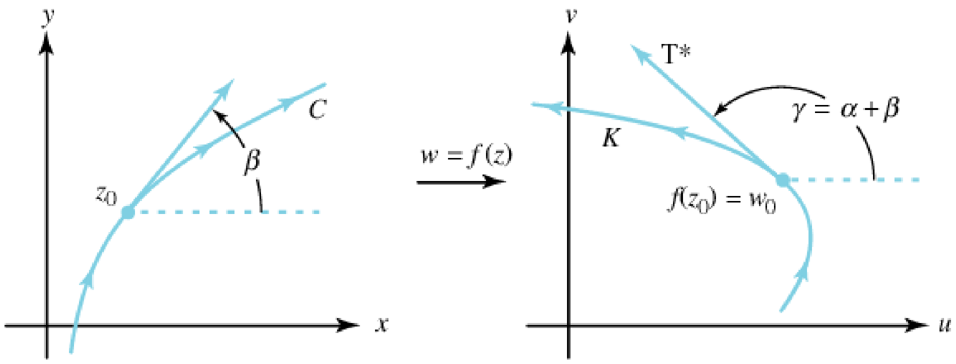


Figure 10.1 The tangents at the points z_0 and w_0, where f(z) is an analytic function and $f'(z_0) \neq 0$.

A mapping $w = f(z)$ is said to be angle preserving, or conformal at z_0, if it preserves angles between oriented curves in magnitude as well as in orientation. Theorem 10.1 shows where a mapping by an analytic function is conformal.

10.6 CONFORMAL MAPPING

Theorem 10.1 (Conformal Mapping). Let f(z) be an analytic function in the domain D, and let z_0 be a point in D. If $f'(z_0) \neq 0$, then f(z) is conformal at z_0.

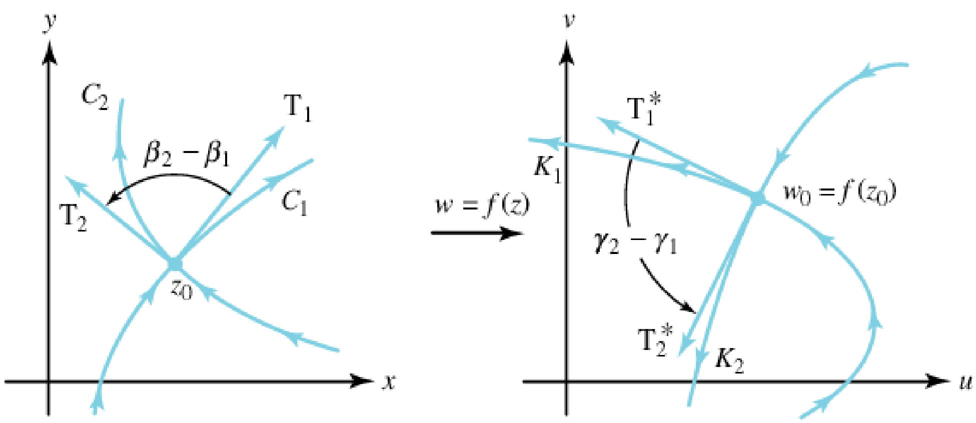
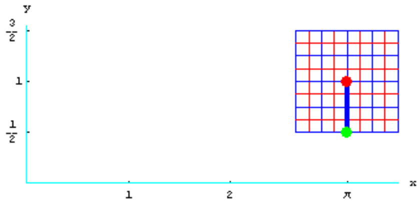
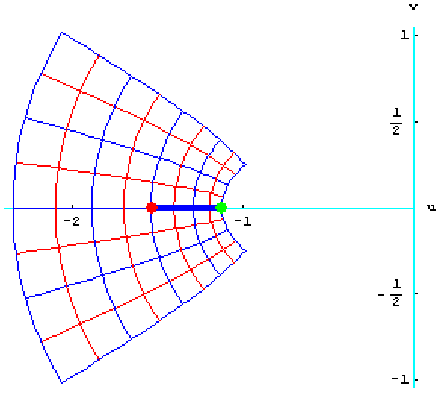


Figure 10.2 The analytic mapping $w = f(z)$ is conformal at the point z_0, where $f'(z_0) \neq 0$.

Example 10.1. Show that the mapping $w = f(z) = \cos(z)$ is conformal at the points $z_1 = i$, $z_2 = 1$, $z_3 = \pi + i$ and, $z_4 = 1 - i$ and determine the angle of rotation given by $\alpha = \mathrm{Arg}(f'(z))$ at the given points.





Solution. Because $f'(z) = -\sin(z)$, we conclude that the mapping $w = \cos(z)$ is conformal at all points except $z = n\pi$, where n is an integer.

Calculation reveals that

$$f'(z_1) = f'(i) = -\sin(i) = -i\sinh 1 \neq 0,$$

$$f'(z_2) = f'(1) = -\sin(1) = -\sin 1 \neq 0,$$

$$f'(z_3) = f'(\pi + i) = -\sin(\pi + i) = i\sinh 1 \neq 0, \quad \text{and}$$

$$f'(z_4) = f'(1 - i) = -\sin(1 - i) = -\cosh 1 \sin 1 + i\cos 1 \sinh 1 \neq 0,$$

Therefore the angle of rotation is given by

$$\alpha_1 = \mathrm{Arg}(f'(z_1)) = \mathrm{Arg}(f'(i)) = \mathrm{Arg}(-i \sinh 1),$$

$$\alpha_1 = -\frac{\pi}{2},$$

$$\alpha_2 = \mathrm{Arg}(f'(z_2)) = \mathrm{Arg}(f'(1)) = \mathrm{Arg}(-\sin 1),$$

$$\alpha_2 = \pi,$$

$$\alpha_3 = \mathrm{Arg}(f'(z_3)) = \mathrm{Arg}(f'(\pi + i)) = \mathrm{Arg}(i \sinh 1),$$

$$\alpha_3 = \frac{\pi}{2}, \text{ and}$$

$$\alpha_4 = \mathrm{Arg}(f'(z_4)) = \mathrm{Arg}(f'(1 - i)) = \mathrm{Arg}(-\cosh 1 \sin 1 + i \cos 1 \sinh 1),$$

$$\alpha_4 = \pi - \arctan(\cot 1 \tanh 1),$$

$$\alpha_4 \approx 2.68677242.$$

Let f(z) be a nonconstant analytic function. If $f'(z_0) = 0$, then z_0 is called a critical point of f(z), and the mapping $w = f(z)$ is not conformal at z_0. The next result shows what happens at a critical point.

Theorem 10.2. Let f(z) be analytic at the point z_0.
If $f'(z_0) = 0, f''(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies angles at the vertex z_0 by the factor k, as shown in Figure 10.3.

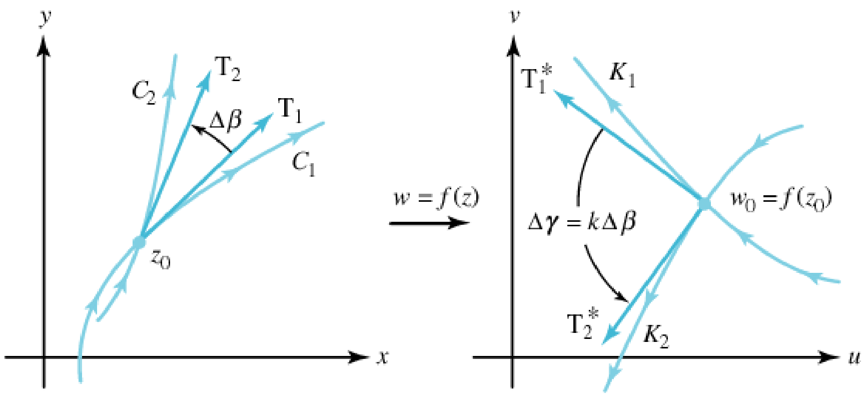


Figure 10.3 The analytic mapping $w = f(z)$ at point z_0, where $f'(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$.

Example 10.2. Show that the mapping $w = f(z) = z^2$ maps the unit square $S = \{x + iy : 0 < x < 1, 0 < y < 1\}$ onto the region in the upper half-plane $\text{Im}(w) > 0$, which lies under the parabolas

$u = 1 - \frac{1}{4}v^2$ and $u = -1 + \frac{1}{4}v^2$ as shown in Figure 10.4.

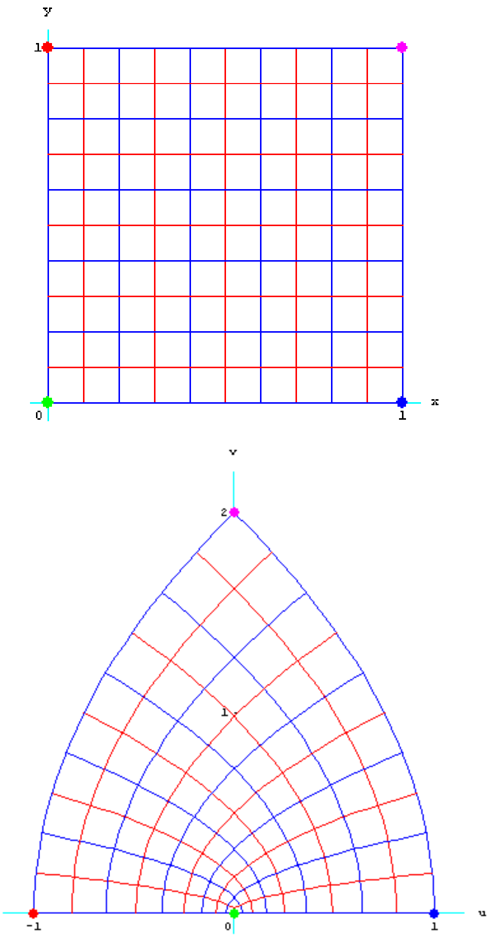
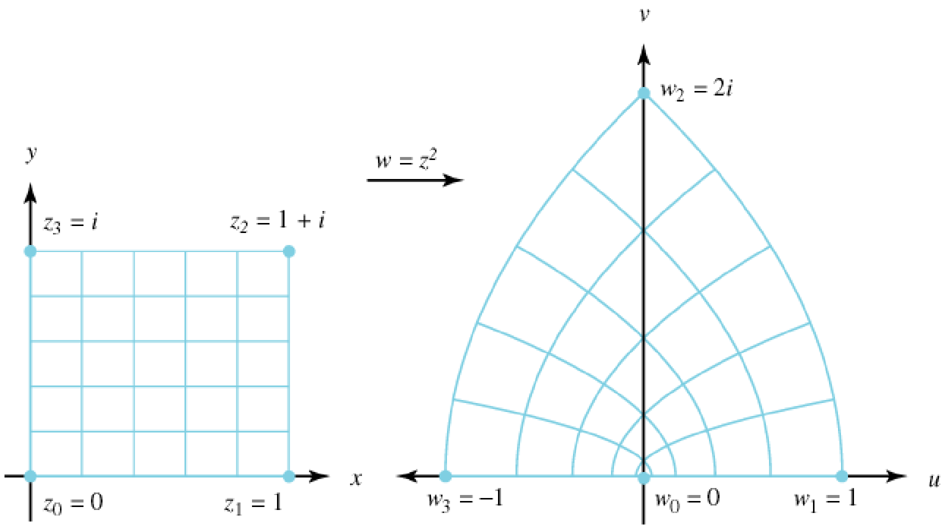


Figure 10.4 The mapping $w = z^2$.

Solution. The derivative is $f'(z) = 2z$, and we conclude that the mapping $w = z^2$ is conformal for all $z \neq 0$. Note that the right angles at the vertices $z_1 = 1$, $z_2 = 1 + i$, and $z_3 = i$ are mapped onto right angles at the vertices $w_1 = 1$, $w_2 = 2i$, and $w_3 = -1$, respectively. At the point $z_0 = 0$, we have $f'(z_0) = f'(0) = 0$ and $f''(z_0) = f''(0) = 2 \neq 0$. Hence angles at the vertex $z_0 = 0$ are magnified by the factor $k = 2$. In particular, the right angle at $z_0 = 0$ is mapped onto the straight angle at $w_0 = 0$.



Another property of a conformal mapping $w = f(z)$ is obtained by considering the modulus of $f'(z_0)$. If z_1 is near z_0, we can use the equation $f(z_1) = f(z_0) + f'(z_0)(z_1 - z_0) + \eta(z_1)(z_1 - z_0)$ and neglect the term $\eta(z_1)(z_1 - z_0)$. We then have the approximation (10-9) $w_1 - w_0 = f(z_1) - f(z_0) \approx f'(z_0)(z_1 - z_0)$.

From Equation (10-9), the distance $|w_1 - w_0|$ between the images of the points z_1 and z_0 given approximately by $|f'(z_0)||z_1 - z_0|$. Therefore we say that the transformation $w = f(z)$ changes small distances near z_0 by the scale factor $|f'(z_0)|$. For example, the scale factor of the transformation $w = f(z) = z^2$ near the point $z_0 = 1 + i$ is $|f'(z_0)| = |2z_0| = |2(1 + i)| = 2\sqrt{2}$.

We also need to say a few things about the inverse transformation $z = g(w)$ of a conformal mapping $w = f(z)$ near a point z_0, where $f'(z_0) \neq 0$. A complete justification of the following assertions relies on theorems studied in advanced calculus.

We express the mapping $w = f(z)$ in the coordinate form (10-10) $u = u(x, y)$ and $v = v(x, y)$.

The mapping in Equations (10-10) represents a transformation from the xy plane into the uv plane, and the Jacobian determinant, $J(x, y)$, is defined by (10-11) $J(x, y) = \begin{vmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{vmatrix}$.

The transformation in Equations (10-10) has a local inverse, provided $J(x, y) \neq 0$. Expanding Equation (10-11) and using the Cauchy--Riemann equations, we obtain

$$J(x_0, y_0) = \begin{vmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{vmatrix}$$

$$= u_x(x_0, y_0) v_y(x_0, y_0) - v_x(x_0, y_0) u_y(x_0, y_0)$$

$$= u_x(x_0, y_0) u_x(x_0, y_0) - v_x(x_0, y_0)(-v_x(x_0, y_0))$$

$$= u_x^2(x_0, y_0) + v_x^2(x_0, y_0)$$

$$= |f'(z_0)|$$

$$\neq 0$$

Consequently, Equations (10-11) and (10-11) imply that a local inverse $z = g(w)$ exists in a neighborhood of the point $w_0 = f(z_0)$. The derivative of g(w) at w_0 is given by the familiar expression

$$g'(w_0) = \lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{w - w_0}$$

$$= \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \lim_{z \to z_0} 1 \Big/ \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

$$= 1 \Big/ \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right)$$

$$= \frac{1}{f'(z_0)}$$

$$= \frac{1}{f'(g(w_0))}$$

2.7 SUMMARY

In this unit we study analytic function and its examples. We study the differentiation of complex analysis. We study the entire function and its examples. We study conformal mapping and its examples. We study rules for differentiation.

2.8 KEYWORD

Conformal: Preserving the correct angles between directions within small areas

Mapping : An operation that associates each element of a given set (the domain) with one or more elements of a second set (the range)

Entire : Not broken, damaged, or decayed

2.9 QUESTIONS FOR REVIEW

Q. 1 Given $f(z) = \cos(z)$, from calculus we know that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$.

Q. 2 Use Formula (3-12) to calculate $\frac{d}{dz}(z^2 + i\,2z + 3)^4$.

Q. 3 Given $f(z) = \sin(z)$, from calculus we know that $f(0) = 0$, $f'(0) = 1$. The Maclaurin polynomial of degree $n = 1$ is $p_1(z) = f(0) + f'(0)z$.

2.10 SUGGESTION READING AND REFERENCES

- Evans, Lawrence C. (1998), Partial Differential Equations, American Mathematical Society.
- Gilbarg, David; Trudinger, Neil, Elliptic Partial Differential Equations of Second Order, ISBN 3-540-41160-7.
- Han, Q.; Lin, F. (2000), Elliptic Partial Differential Equations, American Mathematical Society.
- Jost, Jürgen (2005), Riemannian Geometry and Geometric Analysis (4th ed.), Berlin, New York: Springer-Verlag, ISBN 978-3-540-25907-7.

2.11 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1.2

2 Check in Section 2

Check In Progress-II

Answer Q. 1 Check in section 4

Answer Q 2 Check in Section 3

UNIT 3: APPLICATION OF HARMONIC FUNCTION

STRUCTURE

3.0 OBJECTIVE

The techniques described here require that the vector be mapped to a point in conguration space [17]. The vector acts in Cartesian space, and its conguration can be expressed in terms of joint positions. The path planning problem is then posed as the construction of an obstacle-avoiding path from a start point to a goal point in conguration space (C-space). A bitmap representation of the workspace space for computing the desired harmonic function. Figure 1 illustrates the conversion process: a Cartesian space grid is constructed which contains information about obstacles and goals. Two bits are used for each grid point: one bit designates obstacle points, while the other represents goal regions. Zero is used to denote freespace. Both the obstacle and the goal regions can be arbitrarily shaped up to discretization. C-space bitmaps are rst initialized to 0 (freespace). Then for each C-space bitmap pixel, the equivalent pixels in the Cartesian bitmap are checked using the manipulator forward kinematics. If any of the corresponding Cartesian pixels are occupied, then the C-space bitmap pixel is marked as occupied.

3.1 INTRODUCTION

A wide variety of problems in engineering and physics involve harmonic functions, which are the real or imaginary part of an analytic function. The standard applications are two-dimensional steady-state temperatures, electrostatics, fluid flow, and complex potentials. The techniques of conformal mapping and integral representation can be used to construct a harmonic function with prescribed boundary values. Noteworthy methods include Poisson's integral formulae; the Joukowski transformation; and Schwarz-Christoffel transformation. Modern computer software is capable of implementing these complex analysis methods.

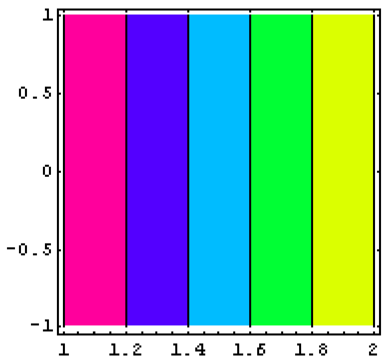
Harmonic functions are solutions to Laplace's equation. Such functions can be used to advantage for potential-eld path planning since they do not exhibit spurious local minima. Harmonic functions are shown here to have a number of properties which are essential to robotics applications. Paths derived from harmonic functions are generally smooth. Harmonic functions also o er, a complete path planning algorithm. We show how a harmonic function can be used as the basis for a reactive admittance control. Such schemes allow incremental updating of the environment model. Methods for computing harmonic functions respond well to sensed changes in the environment and can be used for control while the environment model is being updated. Potential elds were promoted by Khatib [4] for robot path planning. Other authors [5; 6; 7; 8; 9; 10] have used a variety of potential functions for similar purposes. Unfortunately, the usual formulations of potential elds for path planning do not preclude the spontaneous creation of minima other than the goal. The robot can fall into these minima and achieve a stable conguration short of the goal [4; 8; 11; 12; 13]. Koditschek [10] showed that this need not be the case

Notes

in certain types of domains. Connolly, et al. [2] , and independently Akishita, et al. [14] described the application of harmonic functions to the path-planning problem. Harmonic functions are solutions to Laplace's equation. This paper describes harmonic functions and their application to various robot control problems. Harmonic functions are shown here to have several useful properties which make them well suited for robotics applications: Completeness (up to discretization error in the environment model) Robustness in the presence of unanticipated obstacles and errors Ability to exhibit dierent useful modes of behavior Rapid computation (computed as oltages in a resistive grid [15]) In addition, harmonic functions can provide fast surface normal computation and geometric extrapolation. The method described here is a robust form of reactive path planning. Other techniques [13; 16] require substantial o-line computation which prohibits the system from reacting well to unexpected changes in the environment. In contrast, harmonic functions allow the model of the environment to be updated incrementally. Therefore, incomplete or non-stationary environmental models can be modified on-line during execution.

3.2 APPLICATION OF HARMONIC FUNCTION

Example 2.1. Find the function u(x,y) that is harmonic in the vertical strip $a < \mathrm{Re}(z) < b$ and takes on the boundary values $u(a, y) = U_1$ for all y, and $u(b, y) = U_2$ for all y, along the vertical lines $x = a$ and $x = b$, respectively.



Solution. Intuition suggests that we should seek a solution that takes on constant values along the vertical lines of the form $x = x_0$ and that u(x,y) be a function of x alone; that is, $u(x, y) = P(x)$, for $a \le x \le b$ and for all y. Laplace's equation, $u_{xx}(x, y) + u_{yy}(x, y) = 0$, implies that $P''(x) = 0$, which implies $P(x) = m x + c$, where m and c are constants. The stated boundary conditions $u(a, y) = P(a) = U_1$ and $u(b, y) = P(b) = U_2$ lead to the solution $u(x, y) = U_1 + \frac{U_2 - U_1}{b - a}(x - a)$. The level curves $u(x, y) = \text{constant}$ are vertical lines as indicated in Figure 11.1.

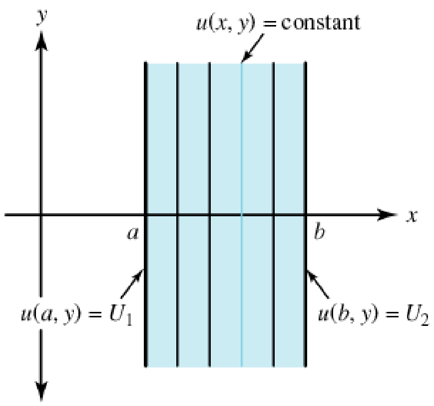
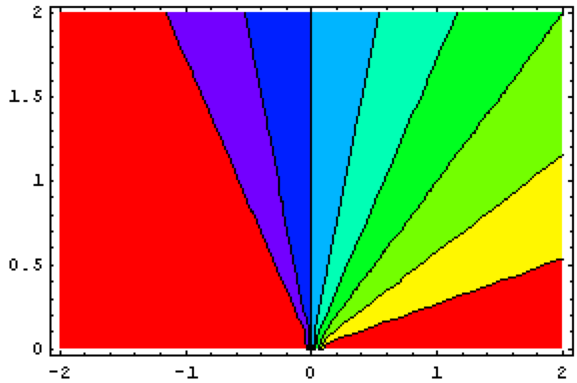


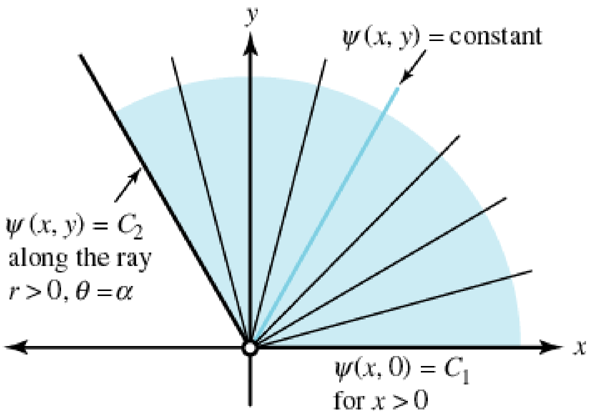
Figure 2.1 Level curves of the harmonic function

$$u(x, y) = U_1 + \frac{U_2 - U_1}{b - a}(x - a).$$

Example 2.2. Find the function $\Phi(x, y)$ that is harmonic in the sector $0 < \operatorname{Arg}(z) < \alpha$ and takes on the boundary values $\Phi(x, 0) = C_1$ for x > 0, $\Phi(x, y) = C_2$ for all points on the ray $r > 0, \theta = \alpha$.



Solution. Recalling that the function $\operatorname{Arg} z$ is harmonic and takes on constant values along rays emanating from the origin, we see that a solution has the form $\Phi(x, y) = a + b \operatorname{Arg} z$, where a and b are constants. The boundary conditions lead to $\Phi(x, y) = C_1 + \frac{C_2 - C_1}{\alpha} \operatorname{Arg} z$. The level curves $\Phi(x, y) = \text{constant}$ are rays emanating from the origin as indicated in Figure 11.2.



<u>**Figure 2.2**</u> Level curves of the harmonic function

$$\Phi(x, y) = C_1 + \frac{C_2 - C_1}{\alpha} \operatorname{Arg} z.$$

A specific example of the general case. Find the function $\Phi(x, y)$ that is harmonic in the sector $0 < \operatorname{Arg}(z) < \frac{2\pi}{3}$ and takes on the boundary values $\Phi(x, 0) = 1$ for x > 0, $\Phi(x, y) = 9$ for all points on the ray $r > 0, \theta = \frac{2\pi}{3}$.

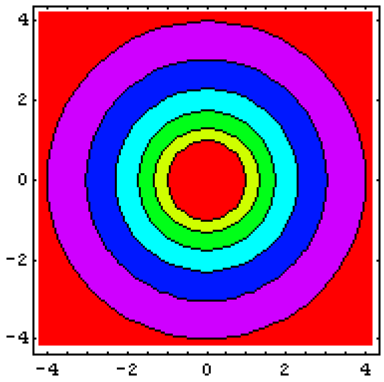
Contour plot for $u[x,y] = 1 + \frac{12 \text{ArcTan}[x, y]}{\pi}$

for $0 < \text{Arg}(z) < \frac{2\pi}{3}$

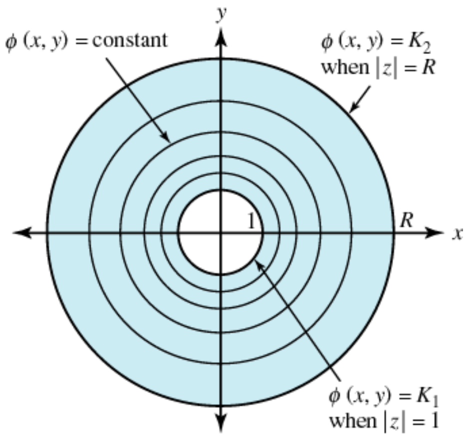
$1 + \frac{12 \text{ArcTan}[x, y]}{\pi} = c$

where c = {2, 3, 4, 5, 6, 7, 8, 9}

Example 2.3. Find the function $\Phi(x, y)$ that is harmonic in the annulus $1 < |z| < R$ and takes on the boundary values $\Phi(x, y) = K_1$ when $|z| = 1$, and $\Phi(x, y) = K_2$ when $|z| = R$.



Solution. This problem is a companion to the one in Example 2.2. Here we use the fact that $\ln|z|$ is a harmonic function, for all $z \neq 0$. The solution is $\Phi(x, y) = K_1 + \dfrac{K_2 - K_1}{\ln R} \ln\left|z\right|$, and the level curves $\Phi(x, y) = \text{constant}$ are concentric circles, as illustrated in Figure 2.3.



<u>**Figure 2.3**</u> Level curves of the harmonic function $\Phi(x, y) = K_1 + \dfrac{K_2 - K_1}{\ln R} \ln\left|z\right|$.

a specific example of the general case. Find the function $\Phi(x, y)$ that is harmonic in the annulus $1 < |z| < 4$ and takes on the boundary values $\Phi(x, y) = 1$ when $|z| = 1$, and $\Phi(x, y) = 6$ when $|z| = 4$.

Notes

Applications involving similar formulas will be found.

$$\text{Contour plot for } u[x,y] = 1 + \frac{5 \operatorname{Log}\left[\sqrt{x^2 + y^2}\right]}{\operatorname{Log}[4]}$$

$$1 + \frac{5 \operatorname{Log}\left[\sqrt{x^2 + y^2}\right]}{\operatorname{Log}[4]} = c$$

where c = {1, 2, 3, 4, 5, 6}

Poisson's Integral Formula for the Upper Half-Plane

The Dirichlet problem for the upper half-plane $\operatorname{Im}(z) > 0$ is to find a function $\phi(x, y)$ that is harmonic in the upper half plane and has the boundary values $\phi(x, 0) = U(x)$, where $U(x)$, is a real-valued function of the real variable x. An important method for solving this problem is our next result which is attributed to the French mathematician Siméon Poisson.

3.3 POISSON'S INTEGRAL FORMULA

Theorem 3.1 (Poisson's Integral Formula). Let U be a real-valued function that is piecewise continuous and bounded for all real t. The function (11-12) $\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{U(t)}{(x - t)^2 + y^2} \, dt$ is harmonic in the upper half plane $\operatorname{Im}(z) > 0$ and has the boundary values $\phi(x, 0) = U(x)$ wherever $U(x)$ is continuous.

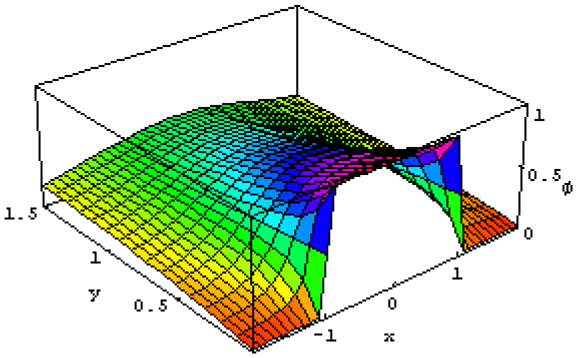
Proof.

Equation (11-12) is easy to determine regarding the Dirichlet problem. Let t1 < t2 < ⋯ < tN denote N points that lie along the x-axis. Let t ∗ 0 < t∗ 1 < ⋯ < t∗ N be N + 1 points chosen so that t ∗ k−1 < tk < t∗ k, for k = 1, 2,..., N, and that U (t) is continuous at each value t ∗ k.

Φ (x, y) = U (t ∗ N) + 1 /π $\sum_{k=1}^{n}$. U t ∗ k−1 − U (t ∗ k) Arg (z − tk) (11-13)

Example 3.1. Find the function $\phi(x, y)$ that is harmonic in the upper half-plane $\mathrm{Im}(z) > 0$, which takes on the boundary values

$$\phi(x, 0) = 1, \quad \text{for} \quad -1 \le x \le 1;$$
$$\phi(x, 0) = 0, \quad \text{for} \quad |x| > 1.$$



Solution. Using Equation (11-12), we obtain

$$\phi(x, y) = \frac{y}{\pi}\int_{-1}^{1} \frac{1}{(x-t)^2 + y^2}\, dt$$

Using techniques from calculus we have the integration formula $\int \frac{y}{(x-t)^2 + y^2}\, dt = \mathrm{Arctan}\left(\frac{y}{x-t}\right)$. We obtain the solution as follows

$$\phi(x, y) = \frac{1}{\pi}\left(\mathrm{Arctan}\,\frac{y}{x-t}\right)\Bigg|_{t=-1}^{t=1}$$

$$= \frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{y}{x+1}\right)$$

$$U[t] = 1$$

$$\phi[x,y] = \frac{y}{\pi}\int_{-1}^{1} \frac{U[t]}{(x-t)^2 + y^2}\, dt$$

$$\phi[x,y] = -\frac{\mathrm{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{\mathrm{ArcTan}\left[\frac{1-x}{y}\right]}{\pi}$$

$$\phi[x,y] = \frac{\mathrm{ArcTan}\left[\frac{y}{-1-x}\right]}{\pi} - \frac{\mathrm{ArcTan}\left[\frac{y}{1-x}\right]}{\pi}$$

Using the trigonometric identity $\arctan(-t) = -\arctan(t)$, the above result can be written as $\phi(x, y) = \frac{1}{\pi}\arctan\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\arctan\left(\frac{y}{x+1}\right)$.

We can verify some of the boundary values by taking limits.

$$\phi[x,y] = -\frac{\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi}$$

$$\lim_{y\to 0} \phi[-2,y] = 0$$

$$\lim_{y\to 0} \phi[\frac{-1}{2},y] = 1$$

$$\lim_{y\to 0} \phi[0,y] = 1$$

$$\lim_{y\to 0} \phi[\frac{1}{2},y] = 1$$

$$\lim_{y\to 0} \phi[2,y] = 0$$

Contour plot for

$$\phi[x,y] = -\frac{\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi}$$

The contours are

$$-\frac{\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} == c$$

where $c = \left\{0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1\right\}$

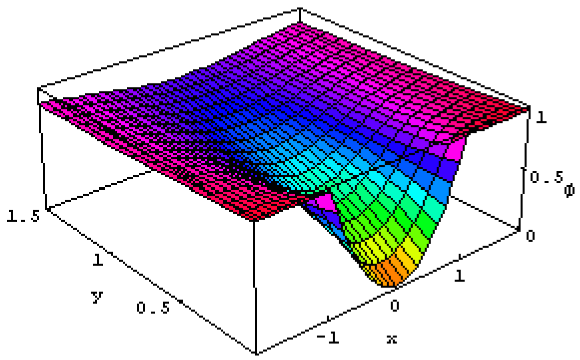
$$\phi[x,y] = -\frac{\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi}$$

Therefore, the function $\phi[x, y] = \frac{1}{\pi}\text{ArcTan}\left[\frac{1-x}{y}\right] + \frac{1}{\pi}\text{ArcTan}\left[\frac{1+x}{y}\right]$ is harmonic in the upper half-plane $\text{Im}[z] > 0$, and takes on the desired boundary values.

Extra Example 1. Find the function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$, which takes on the boundary values

$\phi(x, 0) = x^2$, for $-1 \le x \le 1$;

$\phi(x, 0) = 0$, for $|x| \ge 1$.



Enter the function U[t] and use the Poisson integral to construct $\phi(x, y)$

.

$$U[t] = t^2 \quad \text{for} \quad -1 \le t \le 1$$
$$U[t] = 1 \quad \text{for} \quad |t| \ge 1$$

$$\phi[x,y] = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{U[t]}{(x-t)^2 + y^2} \, dt$$

$$\phi[x,y] = -\frac{1}{2\pi} \left(i \left(4\, i\, y + (-1 + x^2 - y^2) \,\mathrm{Log}\left[1 - \frac{i(-1+x)}{y}\right] + (1 - x^2 + y^2) \,\mathrm{Log}\left[1 + \frac{i(-1+x)}{y}\right] + \mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] - x^2 \,\mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] + y^2 \,\mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] - \mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] + x^2 \,\mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] - y^2 \,\mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] - 2\,\mathrm{Log}\left[-\frac{i}{y}\right] + 2\,\mathrm{Log}\left[\frac{i}{y}\right] + 2\, i\, x\, y \,\mathrm{Log}[(-1+x)^2 + y^2] - 2\, i\, x\, y \,\mathrm{Log}[(1+x)^2 + y^2] \right) \right)$$

$$\phi[x,y] = -\frac{1}{2\pi} \left(i \left(4\, i\, y + (-1 + x^2 - y^2) \,\mathrm{Log}\left[1 - \frac{i(-1+x)}{y}\right] + (1 - x^2 + y^2) \,\mathrm{Log}\left[1 + \frac{i(-1+x)}{y}\right] + \mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] - x^2 \,\mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] + y^2 \,\mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] - \mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] + x^2 \,\mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] - y^2 \,\mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] - 2\,\mathrm{Log}\left[-\frac{i}{y}\right] + 2\,\mathrm{Log}\left[\frac{i}{y}\right] + 2\, i\, x\, y \,\mathrm{Log}[(-1+x)^2 + y^2] - 2\, i\, x\, y \,\mathrm{Log}[(1+x)^2 + y^2] \right) \right)$$

$$\lim_{y \to 0} \phi[-2,y] = 1$$

$$\lim_{y \to 0} \phi\left[\frac{-1}{2},y\right] = \frac{1}{4}$$

$$\lim_{y \to 0} \phi[0,y] = 0$$

$$\lim_{y \to 0} \phi\left[\frac{1}{2},y\right] = \frac{1}{4}$$

$$\lim_{y \to 0} \phi[2,y] = 1$$

Contour plot for

$$\phi[x,y] = -\frac{1}{2\pi} \left(i \left(4\, i\, y + (-1 + x^2 - y^2) \,\mathrm{Log}\left[1 - \frac{i(-1+x)}{y}\right] + (1 - x^2 + y^2) \,\mathrm{Log}\left[1 + \frac{i(-1+x)}{y}\right] + \mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] - x^2 \,\mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] + y^2 \,\mathrm{Log}\left[1 - \frac{i(1+x)}{y}\right] - \mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] + x^2 \,\mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] - y^2 \,\mathrm{Log}\left[1 + \frac{i(1+x)}{y}\right] - 2\,\mathrm{Log}\left[-\frac{i}{y}\right] + 2\,\mathrm{Log}\left[\frac{i}{y}\right] + 2\, i\, x\, y \,\mathrm{Log}[(-1+x)^2 + y^2] - 2\, i\, x\, y \,\mathrm{Log}[(1+x)^2 + y^2] \right) \right)$$

The contours are

$$-\frac{1}{2\pi} (i\ (4\ i\ y + (-1+x^2-y^2)\ \mathrm{Log}[1 - \frac{i(-1+x)}{y}] + (1-x^2+y^2)\ \mathrm{Log}[1 + \frac{i(-1+x)}{y}] + \mathrm{Log}[1 - \frac{i(1+x)}{y}] - x^2\ \mathrm{Log}[1 - \frac{i(1+x)}{y}] + y^2\ \mathrm{Log}[1 - \frac{i(1+x)}{y}] - \mathrm{Log}[1 + \frac{i(1+x)}{y}] + x^2\ \mathrm{Log}[1 + \frac{i(1+x)}{y}] - y^2\ \mathrm{Log}[1 + \frac{i(1+x)}{y}] - 2\ \mathrm{Log}[-\frac{i}{y}] + 2\ \mathrm{Log}[\frac{i}{y}] + 2\ i\ x\ y\ \mathrm{Log}[(-1+x)^2+y^2] - 2\ i\ x\ y\ \mathrm{Log}[(1+x)^2+y^2])) == c$$

where $c = \left\{0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1\right\}$

Notes

$$\phi[x,y] = -\frac{1}{2\pi}\left(i\left(4iy + (-1+x^2-y^2)\,\mathrm{Log}\left[1-\frac{i(-1+x)}{y}\right] + (1-x^2+y^2)\,\mathrm{Log}\left[1+\frac{i(-1+x)}{y}\right] + \right.\right.$$
$$\mathrm{Log}\left[1-\frac{i(1+x)}{y}\right] - x^2\,\mathrm{Log}\left[1-\frac{i(1+x)}{y}\right] + y^2\,\mathrm{Log}\left[1-\frac{i(1+x)}{y}\right] - \mathrm{Log}\left[1+\frac{i(1+x)}{y}\right] + x^2\,\mathrm{Log}\left[1+\frac{i(1+x)}{y}\right] -$$
$$\left.\left. y^2\,\mathrm{Log}\left[1+\frac{i(1+x)}{y}\right] - 2\,\mathrm{Log}\left[-\frac{i}{y}\right] + 2\,\mathrm{Log}\left[\frac{i}{y}\right] + 2ixy\,\mathrm{Log}[(-1+x)^2+y^2] - 2ixy\,\mathrm{Log}[(1+x)^2+y^2]\right)\right)$$

Therefore, the function $\phi[x, y]$ is harmonic in the upper half-plane $\mathrm{Im}[z] > 0$, and takes on the desired boundary values.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 State Poisson's Integral formula.

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

Q. 2 State Poisson's Integral formula for the upper half-plane.

Solution :

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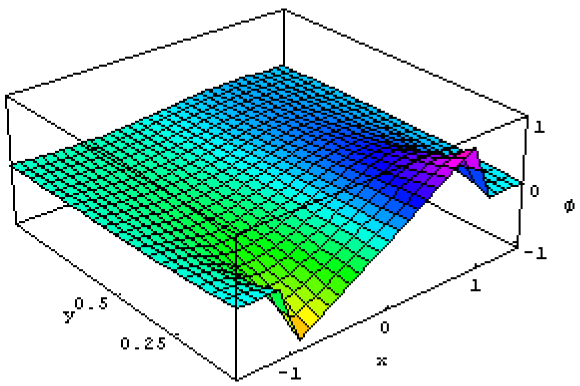
Example 11.12. Find the function $\phi(x, y)$ that is harmonic in the upper half-plane $\mathrm{Im}(z) > 0$, which takes on the boundary values

$\phi(x, 0) = x$, for $-1 < x < 1$;
$\phi(x, 0) = 0$, for $|x| > 1$.



Solution. Using Equation (11-12), we obtain

$$\phi(x,y) = \frac{y}{\pi}\int_{-1}^{1}\frac{t}{(x-t)^2+y^2}\,dt$$

$$= \frac{y}{\pi}\int_{-1}^{1}\frac{(x-t)(-1)}{(x-t)^2+y^2}\,dt + \frac{x}{\pi}\int_{-1}^{1}\frac{y}{(x-t)^2+y^2}\,dt$$

Using techniques from calculus we have the integration formulas

$$\int\frac{(x-t)(-1)}{(x-t)^2+y^2}\,dt = \frac{1}{2}\ln\left((x-t)^2+y^2\right)$$

, and $\int\frac{y}{(x-t)^2+y^2}\,dt = \text{Arctan}\left(\frac{y}{x-t}\right)$. We obtain the solution as follows

$$\phi(x,y) = \frac{y}{\pi}\left(\frac{1}{2}\ln\left((x-t)^2+y^2\right)\right)\Bigg|_{t=-1}^{t=1} + \frac{x}{\pi}\left(\text{Arctan}\left(\frac{y}{x-t}\right)\right)\Bigg|_{t=-1}^{t=1}$$

$$= \frac{y}{2\pi}\ln\left(\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right) + \frac{x}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{x}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right)$$

The function $\phi(x,y)$ is continuous in the upper half-plane, and on the boundary $\phi(x,0)$, except at the discontinuities $x=\pm 1$ on the real axis. The graph in Figure 11.14 shows this phenomenon.

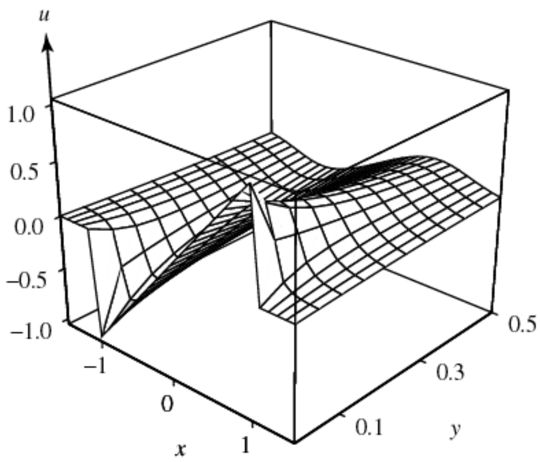


Figure 11.14 The graph of $\phi(x,y)$ with the boundary values $\phi(x,0) = \begin{cases} x, & \text{for } -1 < x < 1; \\ 0, & \text{for } |x| > 1. \end{cases}$

Enter the function U[t] and use the Poisson integral to construct $\phi(x,y)$

.

Notes

$$U[t] = t$$

$$\phi[x,y] = \frac{y}{\pi}\int_{-1}^{1}\frac{U[t]}{(x-t)^2+y^2}\,dt$$

$$\phi[x,y] = -\frac{x\,\mathrm{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi}+\frac{x\,\mathrm{ArcTan}\left[\frac{1-x}{y}\right]}{\pi}+\frac{y\,\mathrm{Log}[1-2x+x^2+y^2]}{2\pi}-\frac{y\,\mathrm{Log}[1+2x+x^2+y^2]}{2\pi}$$

$$\phi[x,y] = \frac{x\,\mathrm{ArcTan}\left[\frac{y}{-1-x}\right]}{\pi}-\frac{x\,\mathrm{ArcTan}\left[\frac{y}{1-x}\right]}{\pi}+\frac{y\,\mathrm{Log}[1-2x+x^2+y^2]}{2\pi}-\frac{y\,\mathrm{Log}[1+2x+x^2+y^2]}{2\pi}$$

Using the identities

$\arctan(-t) = -\arctan(t)$ and $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$, the above result can be written as

$$\phi(x,y) = \frac{x}{\pi}\arctan\left(\frac{y}{x-1}\right)-\frac{x}{\pi}\arctan\left(\frac{y}{x+1}\right)+\frac{y}{2\pi}\ln\left(\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right)$$

. However, for computing values of ArcTan we use the two variable forms of the function and the following version of $\phi[x,y]$. We can verify some of the boundary values by taking limits.

$$\phi[x,y] = \frac{x\,\mathrm{ArcTan}[-1+x,\,y]}{\pi}-\frac{x\,\mathrm{ArcTan}[1+x,\,y]}{\pi}+\frac{y\,\mathrm{Log}\left[\frac{(-1+x)^2+y^2}{(1+x)^2+y^2}\right]}{2\pi}$$

$$\lim_{y\to 0}\phi\left[\frac{-3}{2},y\right] = 0$$

$$\lim_{y\to 0}\phi\left[\frac{-3}{4},y\right] = -\frac{3}{4}$$

$$\lim_{y\to 0}\phi\left[\frac{-1}{2},y\right] = -\frac{1}{2}$$

$$\lim_{y\to 0}\phi[0,y] = 0$$

$$\lim_{y\to 0}\phi\left[\frac{1}{2},y\right] = \frac{1}{2}$$

$$\lim_{y\to 0}\phi\left[\frac{3}{4},y\right] = \frac{3}{4}$$

$$\lim_{y\to 0}\phi\left[\frac{3}{2},y\right] = 0$$

Contour plot for

$$\phi[x,y] = \frac{x\,\mathrm{ArcTan}[-1+x,\,y]}{\pi}-\frac{x\,\mathrm{ArcTan}[1+x,\,y]}{\pi}+\frac{y\,\mathrm{Log}\left[\frac{(-1+x)^2+y^2}{(1+x)^2+y^2}\right]}{2\pi}$$

$$\phi[x,y] = \frac{x\,\mathrm{ArcTan}[-1+x,\,y]}{\pi}-\frac{x\,\mathrm{ArcTan}[1+x,\,y]}{\pi}+\frac{y\,\mathrm{Log}\left[\frac{(-1+x)^2+y^2}{(1+x)^2+y^2}\right]}{2\pi}$$

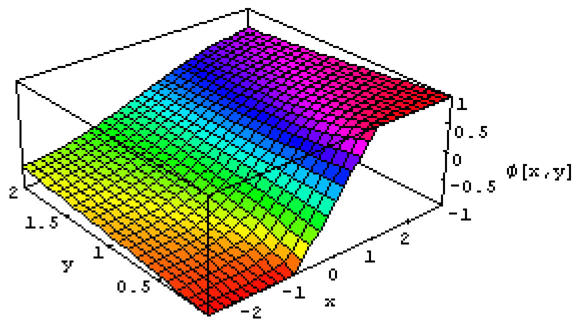
Therefore, the function

$$\phi(x, y) = \frac{x}{\pi} \arctan\left(\frac{y}{x-1}\right) - \frac{x}{\pi} \arctan\left(\frac{y}{x+1}\right) + \frac{y}{2\pi} \ln\left(\frac{(x-1)^2 + y^2}{(x+1)^2 + y^2}\right)$$

is harmonic in the upper half-plane $\mathrm{Im}[z] > 0$, and takes on the desired boundary values.

Example 11.13. Use Poisson's Integral formula to find the harmonic function $\phi(x, y)$ that is harmonic in the upper half-plane $\mathrm{Im}(z) > 0$, that takes on the boundary values

$$\begin{aligned} \phi(x, 0) &= -1, \quad \text{for} \quad x \le -1; \\ \phi(x, 0) &= x, \quad \text{for} \quad -1 \le x \le 1; \\ \phi(x, 0) &= 1, \quad \text{for} \quad 1 \le x. \end{aligned}$$



Solution. Using techniques from Section 11.2, we find that the function

$$v(x, y) = 1 - \frac{1}{\pi} \operatorname{Arctan}\left(\frac{y}{x+1}\right) - \frac{1}{\pi} \operatorname{Arctan}\left(\frac{y}{x-1}\right)$$ is harmonic in

the upper half-plane and has the boundary values

$v(x, 0) = 0$, for $|x| < 1$; $v(x, 0) = -1$, for $x < -1$;

$v(x, 0) = 1$, for $1 < x > 1$. This function can be added to the one in Example 11.12 to obtain the desired result:

$$\begin{aligned} \phi(x, y) &= 1 - \frac{1}{\pi} \operatorname{Arctan}\left(\frac{y}{x-1}\right) - \frac{1}{\pi} \operatorname{Arctan}\left(\frac{y}{x+1}\right) + \frac{y}{2\pi} \ln\left(\frac{(x-1)^2 + y^2}{(x+1)^2 + y^2}\right) \\ &\quad + \frac{x}{\pi} \operatorname{Arctan}\left(\frac{y}{x-1}\right) - \frac{x}{\pi} \operatorname{Arctan}\left(\frac{y}{x+1}\right) \\ &= 1 + \frac{x-1}{\pi} \operatorname{Arctan}\left(\frac{y}{x-1}\right) - \frac{x+1}{\pi} \operatorname{Arctan}\left(\frac{y}{x+1}\right) + \frac{y}{2\pi} \ln\left(\frac{(x-1)^2 + y^2}{(x+1)^2 + y^2}\right) \end{aligned}$$

Figure 11.15 shows the graph of $\phi(x, y)$.

Notes

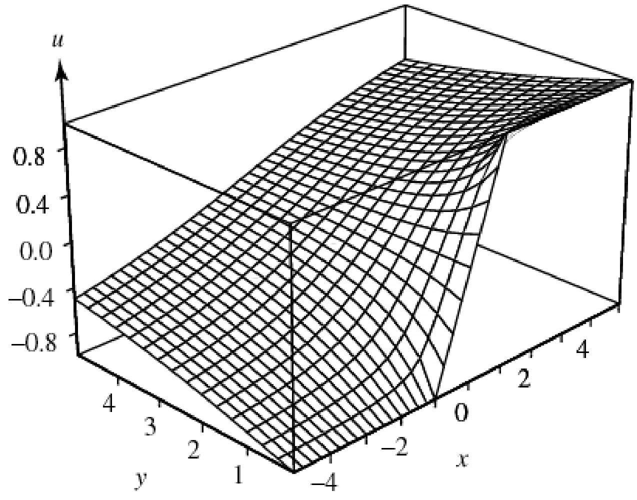


Figure 11.15 The graph of $\phi(x, y)$.

This is similar to Example 11.17, but the method of solution is different. Using techniques from Section 11.2, the function

$v(x, y) = 1 - \frac{1}{\pi}\arctan\left(\frac{y}{x+1}\right) - \frac{1}{\pi}\arctan\left(\frac{y}{x-1}\right)$ is harmonic in the upper half plane and takes on the boundary values

$v(x, 0) = -1$ for $x < -1$,
$v(x, 0) = 0$ for $-1 < x < 1$,
$v(x, 0) = 1$ for $1 < x$. Thus, we should add it to the solution $\phi(x, y)$ in Example 11.12 to obtain the desired result. However, with *Mathematica* we need to use

ArcTan[x ± 1, y] instead of $\text{ArcTan}\left[\frac{y}{x \pm 1}\right]$. Enter the function U[t] and use the Poisson integral to construct $\phi(x, y)$.

U[t] = t

$$V[x,y] = 1 - \frac{\text{ArcTan}[-1+x,\ y]}{\pi} - \frac{\text{ArcTan}[1+x,\ y]}{\pi}$$

$$\phi[x,y] = \frac{y}{\pi}\int_{-1}^{1}\frac{U[t]}{(x-t)^2+y^2}\,dt + V[x,y]$$

$$\phi[x,y] = 1 - \frac{x\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{x\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{\text{ArcTan}[-1+x,\ y]}{\pi} - \frac{\text{ArcTan}[1+x,\ y]}{\pi} + \frac{y\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{y\,\text{Log}[1+2x+x^2+y^2]}{2\pi}$$

$$\phi[x,y] = 1 + \frac{x\,\text{ArcTan}\left[\frac{y}{-1-x}\right]}{\pi} - \frac{x\,\text{ArcTan}\left[\frac{y}{1-x}\right]}{\pi} - \frac{\text{ArcTan}[-1+x,\ y]}{\pi} - \frac{\text{ArcTan}[1+x,\ y]}{\pi} + \frac{y\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{y\,\text{Log}[1+2x+x^2+y^2]}{2\pi}$$

Contour plot for

$$\phi[x,y] = 1 + \frac{x\left(-\text{ArcTan}\left[\frac{-1-x}{y}\right] + \text{ArcTan}\left[\frac{1-x}{y}\right]\right)}{\pi} - \frac{\text{ArcTan}[-1+x,\ y] + \text{ArcTan}[1+x,\ y]}{\pi} + \frac{y\,\text{Log}\left[\frac{1-2x+x^2+y^2}{1+2x+x^2+y^2}\right]}{2\pi}$$

$$\phi[x,y] = 1 + \frac{x\left(-\text{ArcTan}\left[\frac{-1-x}{y}\right] + \text{ArcTan}\left[\frac{1-x}{y}\right]\right)}{\pi} - \frac{\text{ArcTan}[-1+x,\ y] + \text{ArcTan}[1+x,\ y]}{\pi} + \frac{y\,\text{Log}\left[\frac{1-2x+x^2+y^2}{1+2x+x^2+y^2}\right]}{2\pi}$$

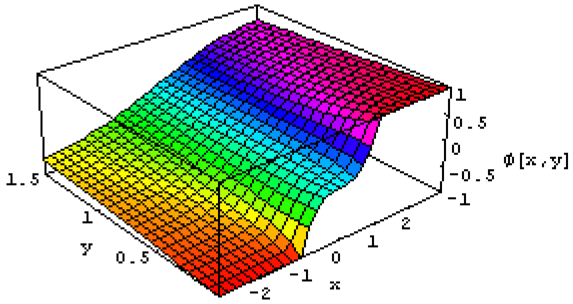
Therefore, the function

$$\phi[x, y] = 1 + \frac{x}{\pi}\left(\mathrm{ArcTan}\left[\frac{1-x}{y}\right] - \mathrm{ArcTan}\left[\frac{-1-x}{y}\right]\right)$$
$$- \frac{1}{\pi}\left(\mathrm{ArcTan}[-1+x, y] + \mathrm{ArcTan}[1+x, y]\right)$$
$$+ \frac{y}{2\pi}\mathrm{Log}\left[\frac{1-2x+x^2+y^2}{1+2x+x^2+y^2}\right]$$

is harmonic in the upper half-plane $\mathrm{Im}[z] > 0$, and takes on the desired boundary values.

Extra Example 2. Use Poisson's Integral formula to find the harmonic function $\phi(x, y)$ that is harmonic in the upper half-plane $\mathrm{Im}(z) > 0$, that takes on the boundary values

$\phi(x, 0) = -1,\quad$ for $\quad x \le -1;$
$\phi(x, 0) = x^2,\quad$ for $\quad -1 \le x \le 1;$
$\phi(x, 0) = 1,\quad$ for $\quad 1 \le x.$



Notes

$$U[t] = -1 \quad \text{for} \quad t \le -1$$

$$U[t] = t^3 \quad \text{for} \quad -1 \le t \le 1$$

$$U[t] = 1 \text{ for } t \ge 1$$

$$V[x,y] = 1 - \frac{\text{ArcTan}[-1+x,\, y]}{\pi} - \frac{\text{ArcTan}[1+x,\, y]}{\pi}$$

$$\phi[x,y] = \frac{y}{\pi}\int_{-1}^{1} \frac{U[t]}{(x-t)^2+y^2}\, dt + V[x,y]$$

$$\phi[x,y] = 1 + \frac{4xy}{\pi} - \frac{x^3\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{3xy^2\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{x^3\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{3xy^2\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{\text{ArcTan}[-1+x,\, y]}{\pi} - \frac{\text{ArcTan}[1+x,\, y]}{\pi} + \frac{3x^2y\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{y^3\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{3x^2y\,\text{Log}[1+2x+x^2+y^2]}{2\pi} + \frac{y^3\,\text{Log}[1+2x+x^2+y^2]}{2\pi}$$

$$\phi[x,y] = 1 + \frac{4xy}{\pi} + \frac{x^3\,\text{ArcTan}\left[\frac{y}{-1-x}\right]}{\pi} - \frac{3xy^2\,\text{ArcTan}\left[\frac{y}{-1-x}\right]}{\pi} - \frac{x^3\,\text{ArcTan}\left[\frac{y}{1-x}\right]}{\pi} + \frac{3xy^2\,\text{ArcTan}\left[\frac{y}{1-x}\right]}{\pi} - \frac{\text{ArcTan}[-1+x,\, y]}{\pi} - \frac{\text{ArcTan}[1+x,\, y]}{\pi} + \frac{3x^2y\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{y^3\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{3x^2y\,\text{Log}[1+2x+x^2+y^2]}{2\pi} + \frac{y^3\,\text{Log}[1+2x+x^2+y^2]}{2\pi}$$

Contour plot for

$$\phi[x,y] = 1 + \frac{4xy}{\pi} - \frac{x^3\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{3xy^2\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{x^3\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{3xy^2\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{\text{ArcTan}[-1+x,\, y]}{\pi} - \frac{\text{ArcTan}[1+x,\, y]}{\pi} + \frac{3x^2y\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{y^3\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{3x^2y\,\text{Log}[1+2x+x^2+y^2]}{2\pi} + \frac{y^3\,\text{Log}[1+2x+x^2+y^2]}{2\pi}$$

$$\phi[x,y] = 1 + \frac{4xy}{\pi} - \frac{x^3\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{3xy^2\,\text{ArcTan}\left[\frac{-1-x}{y}\right]}{\pi} + \frac{x^3\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{3xy^2\,\text{ArcTan}\left[\frac{1-x}{y}\right]}{\pi} - \frac{\text{ArcTan}[-1+x,\, y]}{\pi} - \frac{\text{ArcTan}[1+x,\, y]}{\pi} + \frac{3x^2y\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{y^3\,\text{Log}[1-2x+x^2+y^2]}{2\pi} - \frac{3x^2y\,\text{Log}[1+2x+x^2+y^2]}{2\pi} + \frac{y^3\,\text{Log}[1+2x+x^2+y^2]}{2\pi}$$

Therefore, the function $\phi[x,y]$ is harmonic in the upper half-plane $\text{Im}[z] > 0$, and takes on the desired boundary values.

3.4 THE MEAN-VALUE THEOREM

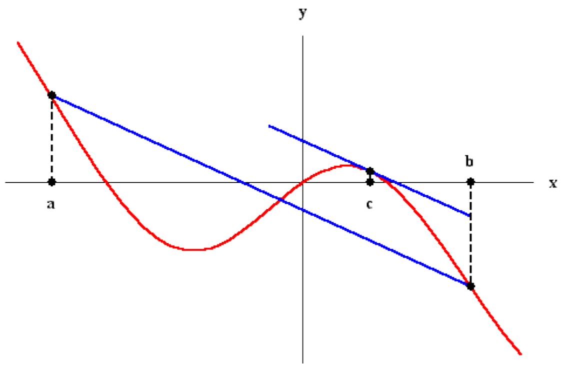
The **Mean Value Theorem** is one of the most important theoretical tools in Calculus. It states that if $f(x)$ is defined and continuous on the interval $[a,b]$ and differentiable on (a,b), then there is at least one number c in the interval (a,b) (that is $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The special case, when $f(a) = f(b)$ is known as **Rolle's Theorem**. In this case, we have $f'(c) = 0$. In other words, there exists a point in the interval (a,b) which has a horizontal tangent. In fact, the Mean Value Theorem can be stated also in terms of slopes. Indeed, the number

$$\frac{f(b)-f(a)}{b-a}$$

is the slope of the line passing through $(a,f(a))$ and $(b,f(b))$. So the conclusion of the Mean Value Theorem states that there exists a point $c \in (a,b)$ such that the tangent line is parallel to the line passing through $(a,f(a))$ and $(b,f(b))$. (see Picture)



Example. Let $f(x) = \frac{1}{x}$, $a = -1$ and $b=1$. We have

$$\frac{f(b)-f(a)}{b-a} = \frac{2}{2} = 1.$$

On the other hand, for any $c \in (-1,1)$, not equal to 0, we have

$$f'(c) = -\frac{1}{c^2} \neq 1.$$

So the equation $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

does not have a solution in c. This does **not** contradict the Mean Value Theorem, since $f(x)$ is not even continuous on $[-1,1]$.

Notes

Remark. It is clear that the derivative of a constant function is 0. But you may wonder whether a function with derivative zero is constant. The answer is yes. Indeed, let $f(x)$ be a differentiable function on an interval I, with $f'(x) = 0$, for every $x \in I$. Then for any a and b in I, the Mean Value Theorem implies

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c between a and b. So our assumption implies

$$f(b) - f(a) = 0 \cdot (b - a) = 0.$$

Thus $f(b) = f(a)$ for any a and b in I, which means that $f(x)$ is constant.

Exercise 1. Show that the equation

$$2x^3 + 3x^2 + 6x + 1 = 0$$

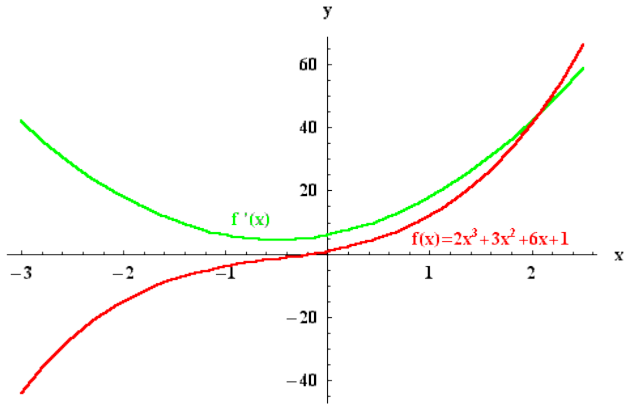
has exactly one real root.

Let $f(x) = 2x^3 + 3x^2 + 6x + 1$. We have $f(0)=1$ and $f(-1) = -4$. So the Intermediate Value Theorem shows that there exists a point c between -1 and 0 such that $f(c) = 0$. Consequently our equation has at least one real root.

Let us now show that this equation has also at most one real root. Assume not, then there must exist at least two roots c_1 and c_2, with $c_1 < c_2$. Then we have $f(c_1) = 0$ and $f(c_2) = 0$. Rolle's Theorem implies the existence of a point c between c_1 and c_2 such that

$$f'(c) = 6c^2 + 6c + 6 = 0.$$

But the quadratic equation $6c^2 + 6c + 6 = 0$ does not have real roots, yielding a contradiction to our assumption that $f(x)$ had at least two roots. Conclusion: our original equation has exactly one real root.



Exercise 2. Show that

$$|\cos(2a) - \cos(2b)| \le 2|a-b|$$

for all real numbers *a* and *b*. Try to find a more general statement.

Answer to Exercise 2

Set $f(x) = \cos(2x)$. Then we have $f'(x) = -2\sin(2x)$. The inequality clearly holds when *a*=*b*. For any numbers *a* and *b*, with $a \ne b$, the Mean Value Theorem implies

$$\frac{f(b) - f(a)}{b-a} = \frac{\cos(2b) - \cos(2a)}{b-a} = -2\sin(2c)$$

for some *c* between *a* and *b*. Since $|\sin(2c)| \le 1$, we get

$$\left|\frac{\cos(2b) - \cos(2a)}{b-a}\right| \le 2$$

which obviously gives the desired inequality. In general, if *f*(*x*) is a differentiable function on an interval *I* with $|f'(x)| \le M$ for any $x \in I$, then we have

$$|f(x_1) - f(x_2)| \le M|x_1 - x_2|$$

for any $x_1, x_2 \in I$.

Example 3 Suppose that we know that f(x) is continuous and differentiable on [6,15] Let's also suppose that we know that f(6)=−2f(6)=−2 and that we know that f′(x)≤10. What is the largest possible value for f(15)?

Let's start with the conclusion of the Mean Value Theorem.

f(15)−f(6)=f′(c)(15−6)

Plugging in for the known quantities and rewriting this a little gives,

f(15)=f(6)+f′(c)(15−6)=−2+9f′(c)

Now we know that f′(x)≤10 so in particular we know that f′(c)≤10. This gives us the following,

f(15)=−2+9f′(c)≤−2+(9)10=88

All we did was replace f′(c) with its largest possible value.

This means that the largest possible value for f(15) is 8.

Example 4 Suppose that we know that f(x) is continuous and differentiable everywhere. Let's also suppose that we know that f(x) has two roots. Show that f′(x) must have at least one root.

It is important to note here that all we can say is that f′(x) will have at

least one root. We can't say that it will have exactly one root. So don't confuse this problem with the first one we worked on.

This is actually a fairly simple thing to prove. Since we know that f(x) has two roots let's suppose that they are aa and b Now, by assumption we know that f(x)f(x) is continuous and differentiable everywhere and so, in particular, it is continuous on [a,b] and differentiable on (a,b).

Therefore, by the Mean Value Theorem there is a number cc that is between aa and bb (this isn't needed for this problem, but it's true so it should be pointed out) and that,

$f'(c)=f(b)-f(a)b-a$

But we now need to recall that aa and bb are roots of f(x)f(x) and so this is,

$f'(c)=0-0b-a=0$

Or, $f'(x)$ has a root at $x=c$

Again, it is important to note that we don't have a value of c. We have

only shown that it exists. We also haven't said anything about c being the only root. It is completely possible for $f'(x)$ to have more than one root.

It is completely possible to generalize the previous example significantly. For instance if we know that f(x) is continuous and differentiable everywhere and has three roots we can then show that not only will $f'(x)$ have at least two roots but that $f''(x)$ will have at least one root. We'll leave it to you to verify this, but the ideas involved are identical to those in the previous example.

We'll close this section out with a couple of nice facts that can be proved using the Mean Value Theorem. Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval [a,b].

Fact. If f′(x)=0f′(x)=0 for all xx in an interval (a,b)(a,b) then f(x)f(x) is constant on (a,b)(a,b).

This fact is very easy to prove so let's do that here.

First, notice that because we are assuming the derivative exists on (a,b) we know that f(x) is differentiable on (a,b). In addition, we **know** that if a function is differentiable on an interval then it is also continuous on that interval and so f(x) will also be continuous on (a,b).

Now, take any two x's in the interval (a,b), say x1 and x2. Then since f(x) is continuous and differentiable on (a,b) it must also be continuous and differentiable on [x1,x2]. This means that we can apply the Mean Value Theorem for these two values of x. Doing this gives,

$$f(x2)-f(x1)=f'(c)(x2-x1)$$

where x1<c<. But by assumption f′(x)=0 for all x in an interval (a,b) and

so, in particular, we must have,

$$f'(c)=0$$

Putting this into the equation above gives,

$$f(x2)-f(x1)=0 \Rightarrow f(x2)=f(x1)$$

Now, since x1 and x2 where any two values of x in the

interval (a,b) we can see that we must have f(x2)=f(x1) for all x1 and x2 in the interval and this is exactly what it means for a function to be constant on the interval and so we've proven the fact.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 Define Mean Value Theorem.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q. 2 Give statement of mean value theorem.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

3.5 MAXIMUM AND MINIMUM PRINCIPAL

In mathematics, the **maximum modulus principle** in complex analysis states that if *f* is a holomorphic function, then the modulus $|f|$ cannot exhibit a true local maximum that is properly within the domain of *f*.

In other words, either *f* is a constant function, or, for any point z_0 inside the domain of *f* there exist other points arbitrarily close to z_0 at which $|f|$ takes larger values.

Formal statement

Let *f* be a function holomorphic on some connected open subset *D* of the complex plane $\mathbb{C}$ and taking complex values. If z_0 is a point in *D* such that

$$|f(Z)| \geq |f(Zo)|$$

for all *z* in a neighborhood of z_0, then the function *f* is constant on *D*.

By switching to the reciprocal, we can get the **minimum modulus principle**. It states that if *f* is holomorphic within a bounded domain *D*, continuous up to the boundary of *D*, and non-zero at all points, then $|f(z)|$ takes its minimum value on the boundary of *D*.

Alternatively, the maximum modulus principle can be viewed as a special case of the open mapping theorem, which states that a nonconstant holomorphic function maps open sets to open sets. If $|f|$ attains a local maximum at *z*, then the image of a sufficiently small open neighborhood of *z* cannot be open. Therefore, *f* is constant.

1. U¯ is compact because U is bounded, hence ff attains its maximum on U¯. If ff is non-constant then it is an open mapping, hence does not have a local maximum in U. Therefore

$$f(z)|\leq \max\{|f(w)|:w\in\partial U\}$$

for all z∈U, since this inequality certainly holds if ff is constant.

2. If f(w)=0f(w)=0 for some w∈∂Uw∈∂U then the inequality

$$0|f(z)|<\min\{|f(w)|:w\in\partial U\}=0$$

is impossible, so we may assume that ff does not vanish on ∂U. If f has no zeros in U, then 1f is holomorphic in U and continuous on U¯, and it follows from part (1) applied to 1f that

$$f(z)|\geq \min\{|f(w)|:w\in\partial U\}$$

for all $z\in U$. Therefore $f(z)|<\min\{|f(w)|:w\in\partial U\}$ for some $z\in U$, then f must have a zero in U.

Using Gauss's mean value theorem

Another proof works by using Gauss's mean value theorem to "force" all points within overlapping open disks to assume the same value. The disks are laid such that their centers form a polygonal path from the value where $f(z)$ is maximized to any other point in the domain, while being totally contained within the domain. Thus the existence of a maximum value implies that all the values in the domain are the same, thus $f(z)$ is constant.

Physical interpretation

A physical interpretation of this principle comes from the heat equation. That is, since $\log |f(z)|$ is harmonic, it is thus the steady state of a heat flow on the region D. Suppose a strict maximum was attained on the interior of D, the heat at this maximum would be dispersing to the points around it, which would contradict the assumption that this represents the steady-state of a system.

Major results

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, as is stated by the Cauchy integral theorem. The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary (as shown in Cauchy's integral formula). Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is applicable (see methods of contour integration). A "pole" (or isolated singularity) of a function is a point where the function's value becomes unbounded, or "blows up". If a function has such a pole, then one can compute the function's residue there, which can be used to compute path integrals involving the function; this is the content of the powerful residue theorem. The remarkable behavior of holomorphic functions near essential singularities is described by Picard's Theorem. Functions that have only poles but no essential singularities are called monomorphic. Laurent series are the complex-valued equivalent to Taylor series but can be used to study the behavior of functions near singularities through infinite sums of better-understood functions, such as polynomials.

A bounded function that is holomorphic in the entire complex plane must be constant; this is Lowville's. It can be used to provide natural and short proof for the fundamental theorem of algebra which states that the field of complex numbers is algebraically closed.

If a function is holomorphic throughout a connected domain then its values are fully determined by its values on any smaller subdomain. The function on the larger domain is said to be analytically continued from its values on the smaller domain. This allows the extension of the definition of functions, such as the Riemann zeta function, which are initially defined in terms of infinite sums that converge only on limited domains to almost the entire complex plane. Sometimes, as in the case of the natural logarithm, it is impossible to analytically continue a holomorphic function to a non-simply connected domain in the complex plane but it is possible to extend it to a holomorphic function on a closely related surface known as a Riemann surface.

All this refers to complex analysis in one variable. There is also a very rich theory of complex analysis in more than one complex dimension in which the analytic properties such as power series expansion carry over whereas most of the geometric properties of holomorphic functions in one complex dimension (such as conformity) do not carry over. The Riemann mapping theorem about the conformal relationship of certain domains in the complex plane, which may be the most important result in the one-dimensional theory, fails dramatically in higher dimensions.

A major use of certain complex spaces is in quantum mechanics as wave functions.

Application

- The fundamental theorem of algebra.
- Schwarz's lemma, a result which in turn has many generalizations and applications in complex analysis.
- The Phragmén–Lindelöf principle, an extension to unbounded domains.
- The Borel–Carathéodory theorem, which bounds an analytic function in terms of its real part.
- The Hadamard three-lines theorem, a result of the behavior of bounded holomorphic functions on a line between two other parallel lines in the complex plane.

3.6 SUMMARY

In this unit, we study the application of harmonic function and its properties with examples. We study Poisson's Integral Formula and Poisson's Integral formula for the upper half-plane. We study minimal and maximum principal and its properties.

3.7 KEYWORD

Holomorphic : In mathematics, a *holomorphic* function is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighborhood of the point

Plane : A flat surface on which a straight line joining any two points on it would wholly lie

Conformal : Preserving the correct angles between directions within small areas (though distorting distances).

3.8 QUESTIONS FOR REVIEW

Q. 1 If f′(x)=0 f′(x)=0 for all x in an interval (a,b) then f(x) is constant on (a,b).

Q. 2 If f′(x)=g′(x) for all xx in an interval (a,b) then in this interval we have f(x)=g(x)+c where c is some constant.

Q. 3 Theorem:(Minimum Principle) Let D be a domain and $u : D \to \mathbb{R}$ be continuous and satisfy the MVP on D. If $\exists \alpha \in D$ such that $u(z) \geq u(\alpha)$, then u is a constant function.

Q. 4 Let D be a domain and $u : D \to \mathbb{R}$ be harmonic function on D. If $\exists \alpha \in D$ such that $u(z) \geq u(\alpha)$, then u is a constant function.

Q. 5 Every harmonic function on a domain have the MVP

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3.10 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 2

2 Check in Section 2

Check In Progress-II

Answer Q. 1 Check in section 4

Answer Q 2 Check in Section 4

UNIT 4: THE DIRICHLET PROBLEM FOR THE UNIT DISK AND FOURIER SERIES

STRUCTURE

4.0 OBJECTIVE

- Learn Dirichlet Problem For the Unit Disk
- Learn solution of Dirichlet Problem
- Learn Fourier Series
- Working with example of Fourier series
- Learn Characterization of mean value property

4.1 INTRODUCTION: DIRICHLET PROBLEM

In mathematics, a **Dirichlet problem** is the problem of finding a function which solves a specified partial differential equation (PDE) in the interior of a given region that takes prescribed values on the boundary of the region.

The Dirichlet problem can be solved for many PDEs, although originally it was posed for Laplace's equation. In that case the problem can be stated as follows:

Given a function f that has values everywhere on the boundary of a region in $\mathbf{R}^n$, is there a unique continuous function u twice continuously differentiable in the interior and continuous on the boundary, such that u is harmonic in the interior and $u = f$ on the boundary?

This requirement is called the Dirichlet boundary condition. The main issue is to prove the existence of a solution; uniqueness can be proved using the maximum principle.

4.1.2 History

The Dirichlet problem goes back to George Green who studied the problem on general domains with general boundary conditions in his Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, published in 1828. He reduced the problem into a problem of constructing what we now call Green's functions, and argued that Green's function exists for any domain. His methods were not rigorous by today's standards, but the ideas were highly influential in the subsequent developments. The next steps in the study of the Dirichlet's problem were taken by Karl Friedrich Gauss, William Thomson (Lord Kelvin) and Peter Gustav Lejeune Dirichlet after whom the problem was named and the solution to the problem (at least for the ball) using the Poisson kernel was known to Dirichlet (judging by his 1850 paper submitted to the Prussian academy). Lord Kelvin and Dirichlet suggested a solution to the problem by a variational method based on the minimization of "Dirichlet's energy". According to Hans Freudenthal (in the Dictionary of Scientific Biography, vol 11), Bernhard Riemann was the first mathematician who solved this variational problem based on a method which he called Dirichlet's principle. The existence of a unique solution is very plausible by the 'physical argument': any charge distribution on the boundary should, by the laws of electrostatics, determine an electrical potential as solution. However, Karl Weierstrass found a flaw in Riemann's argument, and a rigorous proof of existence was found only in 1900 by David Hilbert, using his direct method in the calculus of variations. It turns out that the existence of a solution depends delicately on the smoothness of the boundary and the prescribed data.

4.2 METHODS OF SOLUTION

For bounded domains, the Dirichlet problem can be solved using the Perron method, which relies on the maximum principle for sub harmonic functions. This approach is described in many text books. It is not well-suited to describing smoothness of solutions when the boundary is smooth. Another classical Hilbert space approach through Sobolev spaces does yield such information.[2] The solution of the Dirichlet problem using Sobolev spaces for planar domains can be used to prove the smooth version of the Riemann mapping theorem. Bell (1992) has outlined a different approach for establishing the smooth Riemann mapping theorem, based on the reproducing kernels of Szegő and Bergman, and in turn used it to solve the Dirichlet problem. The classical methods of potential theory allow the Dirichlet problem to be solved directly in terms of integral operators, for which the standard theory of compact and Freehold is applicable. The same methods work equally for the Neumann problem.

Notes

The Dirichlet Problem for the Unit Disk

We have emphasized practical applications of harmonic functions to steady state temperatures, electrostatics. In Section 1.2 we introduced the N-Value Dirichlet problem, and showed how to solve Laplace's equation $\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$, in the upper-half plane, for a harmonic function $u(x, y)$ that has certain specified boundary values on the real axis. Now we will develop the solution to the Dirichlet problem in the closed unit disk $\overline{D_1}(0) = \{z : |z| \le 1\} = \{re^{i\theta} : 0 \le r \le 1\}$.

Dirichlet Problem for the Unit Disk Given a real valued function $U(\theta)$ that is both piecewise continuous and a bounded function. Let $U(\theta)$ be considered as boundary values on the unit circle $C_1(0) = \{z : |z| = 1\}$, in the sense that (1.0) $u(\cos\theta, \sin\theta) = U(\theta)$ for $-\pi \le \theta \le \pi$, at points $z = \cos\theta + i\sin\theta$ on the unit circle. The Dirichlet problem for the closed unit disk $\overline{D_1}(0) = \{z : |z| \le 1\}$ is to extend $u(\cos\theta, \sin\theta)$ to be $u(r\cos\theta, r\sin\theta) = u(x, y)$ for $x + iy = re^{i\theta} \in D_1(0) = \{z : |z| < 1\}$, where $u(x, y)$ is harmonic in the unit disk and take on the boundary values (1.1) at points where $U(\theta)$ is continuous.

Our first method of solution uses the Fourier Series representation for $U(\theta)$

4.2.1 Extended Fourier Series in the unit disk

Theorem 2.1 (Extended Fourier Series in the unit disk). Let $U(\theta)$ be the boundary values on the unit circle $C_1(0) = \{z : |z| = 1\}$, (1) $u(\cos\theta, \sin\theta) = U(\theta)$ for $-\pi \le \theta \le \pi$. If $U(\theta)$ has the Fourier series representation $u(\cos\theta, \sin\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))$

, then (2)

$$u(r\cos\theta, r\sin\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta))$$, solves the Dirichlet problem and is a harmonic function in the unit disk $D_1(0) = \{z : |z| < 1\}$.

We can motivate if we are allowed to make the following claim: The series in (1.1) takes on the boundary values in Equation (1.1) at points on $C_1(0)$ where the radial limits exist, that is

$$\lim_{r \to 1} u(r\cos\theta, r\sin\theta) = \lim_{r \to 1}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta))\right)$$

$$= \frac{a_0}{2} + \lim_{r \to 1}\sum_{n=1}^{\infty}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta))$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\lim_{r \to 1}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta))$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n 1^n \cos(n\theta) + b_n 1^n \sin(n\theta))$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$= u(\cos\theta, \sin\theta)$$

Furthermore, Exercise 12 in Section 3.3 shows that each of the terms, $a_n r^n \cos(n\theta)$ and $b_n r^n \sin(n\theta)$ are harmonic, and so it is reasonable to conclude that the infinite series representing $u(r\cos\theta, r\sin\theta)$ in Equation (2) will be harmonic. Remark. The radial limits will exist except at the finite number of points where $U(\theta)$ is discontinuous. There are details regarding the convergence of the series and the existence of radial limits that are left for advanced study.

Example 2.2. Find the function $u(x, y) = u(r\cos\theta, r\sin\theta)$ that is harmonic in the unit disk $D_1(0) = \{z : |z| < 1\}$ and takes on the boundary values $u(\cos\theta, \sin\theta) = U(\theta) = \frac{\theta}{2}$ for $-\pi < \theta < \pi$.

Solution.

Notes

we write the Fourier series for $U(\theta)$

$$u(\cos\theta, \sin\theta) = \sum_{n=1}^{\infty} \frac{-\cos(n\pi)}{n} \sin(n\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$$
, .

for the extended Fourier series solution of the Dirichlet problem, we obtain

$$u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{-\cos(n\pi)}{n} r^n \sin(n\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$$
.

The series representations of $u(r\cos\theta, r\sin\theta)$ take on the prescribed boundary values at points where $U(\theta)$ is continuous. The boundary function $U(\theta)$ is discontinuous at $z = -1$, which corresponds to $\theta = \pm\pi$; which are points where $U(\theta)$ was not prescribed. The approximations

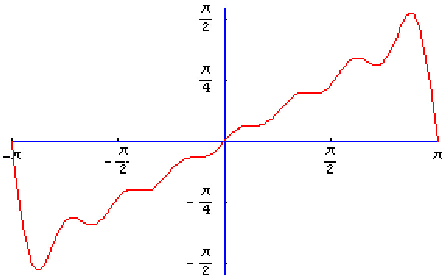
$$U(\theta) \approx U_7(\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} \sin(n\theta)$$
and

$$u(r\cos\theta, r\sin\theta) \approx u_7(r\cos\theta, r\sin\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$$
, and

the true

solutions $$U(\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} \sin(n\theta)$$ and

$$u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$$
are shown in the figures below.



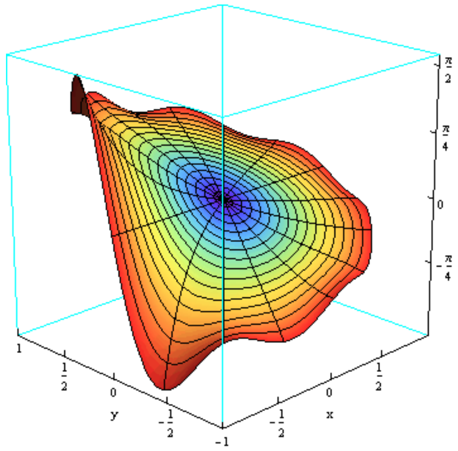
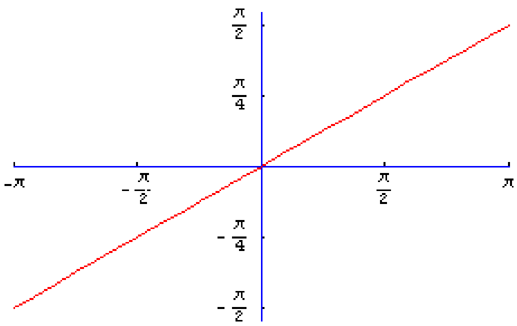
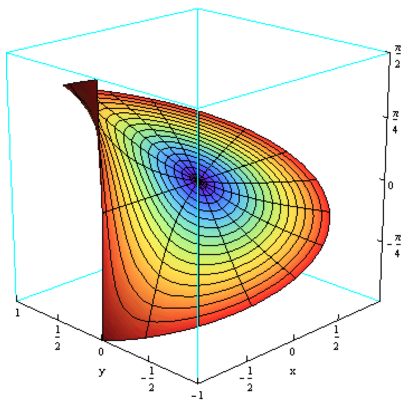


Figure A. The functions $$U_7(\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} \sin(n\theta)$$

and $$u_7(r\cos\theta, r\sin\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$$.





Notes

Figure.B. The functions $U(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$

and $u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$.

we have the Fourier series for $U(\theta) = \frac{\theta}{2}$

$$u(\cos\theta, \sin\theta) = \sum_{n=1}^{\infty} \frac{-\cos(n\pi)}{n} \sin(n\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$$
, .

for the extended Fourier series solution of the Dirichlet problem, we obtain

$$u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{-\cos(n\pi)}{n} r^n \sin(n\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$$
.

4.2.2 An approximation using a partial sum.

Summing up the first seven terms we get the approximations $U_7(\theta)$ and $u_7(r\cos\theta, r\sin\theta)$.

$$\text{"}\mathbf{U_7(\theta)}\text{"} == \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} \mathrm{Sin}[n\theta]$$

$$U_7(\theta) == \mathrm{Sin}[\theta] - \frac{1}{2}\mathrm{Sin}[2\theta] + \frac{1}{3}\mathrm{Sin}[3\theta] - \frac{1}{4}\mathrm{Sin}[4\theta] + \frac{1}{5}\mathrm{Sin}[5\theta] - \frac{1}{6}\mathrm{Sin}[6\theta] + \frac{1}{7}\mathrm{Sin}[7\theta]$$

$$\text{"}\mathbf{u_7(r\ cos\ \theta, r\ sin\ \theta)}\text{"} == \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^n \mathrm{Sin}[n\theta]$$

$$u_7(r\ \cos\ \theta, r\ \sin\ \theta) == r\,\mathrm{Sin}[\theta] - \frac{1}{2} r^2 \mathrm{Sin}[2\theta] + \frac{1}{3} r^3 \mathrm{Sin}[3\theta] - \frac{1}{4} r^4 \mathrm{Sin}[4\theta] + \frac{1}{5} r^5 \mathrm{Sin}[5\theta] - \frac{1}{6} r^6 \mathrm{Sin}[6\theta] + \frac{1}{7} r^7 \mathrm{Sin}[7\theta]$$

The functions $U_7(\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} \sin(n\theta)$

and $u_7(r\cos\theta, r\sin\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$.

Summing up all of the terms we get the boundary value function $U(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$ on the unit circle $C_1(0) = \{z : |z| = 1\}$, and the harmonic function $u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta)$ in the unit disk $D_1(0) = \{z : |z| < 1\}$.

$$\text{"}\mathbf{U(\theta)}\text{"} == \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathbf{Sin[n\,\theta]}$$

$$U(\theta) == \frac{1}{2} i \left(\text{Log}\left[1 + e^{-i\theta}\right] - \text{Log}\left[1 + e^{i\theta}\right]\right)$$

$$\text{"}\mathbf{u(r\ cos\ \theta, r\ sin\ \theta)}\text{"} == \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \mathbf{Sin[n\,\theta]}$$

$$u(r\ \cos\ \theta, r\ \sin\ \theta) == \frac{1}{2} i \left(\text{Log}\left[1 + e^{-i\theta} r\right] - \text{Log}\left[1 + e^{i\theta} r\right]\right)$$

Aside. The Maple commands are similar

$$> \text{expand}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(n\,t)\right)$$

$$-\frac{1}{2} I \ln(1 + e^{It}) + \frac{1}{2} I \ln(1 + e^{-It})$$

$$> \text{expand}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} r^n \sin(n\,t)\right)$$

$$-\frac{1}{2} I \ln(1 + r\, e^{It}) + \frac{1}{2} I \ln(1 + r\, e^{-It})$$

We can use *Mathematica* to plot the boundary function $U(\theta)$ and harmonic function $u(r\cos\theta, r\sin\theta)$.

The boundary function

$$U(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta) = \frac{1}{2} i \left(\text{Log}\left[1 + e^{-i\theta}\right] - \text{Log}\left[1 + e^{i\theta}\right]\right),$$

and the harmonic function

$$u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(n\theta) = \frac{1}{2} i \left(\text{Log}\left[1 + e^{-i\theta} r\right] - \text{Log}\left[1 + e^{i\theta} r\right]\right)$$

when we sum the infinite series we get solutions involving the logarithm function.

Remark 2. the other form of the solution will be similar.

$$\text{"}U(\theta)\text{"} == \sum_{n=1}^{\infty} \frac{-\mathrm{Cos}[n\pi]}{n} \mathrm{Sin}[n\theta]$$

$$U(\theta) == \frac{1}{2} i \left(\mathrm{Log}\left[1 + e^{-i\theta}\right] - \mathrm{Log}\left[1 + e^{i\theta}\right]\right)$$

$$\text{"}u(r \cos\theta, r \sin\theta)\text{"} == \sum_{n=1}^{\infty} \frac{-\mathrm{Cos}[n\pi]}{n} r^n \mathrm{Sin}[n\theta]$$

$$u(r \cos\theta, r \sin\theta) == \frac{1}{2} i \left(\mathrm{Log}\left[1 + e^{-i\theta} r\right] - \mathrm{Log}\left[1 + e^{i\theta} r\right]\right)$$

Poisson's integral formula for the upper half-plane, that was introduced It shows that the value of the harmonic function $u(r\cos\theta, r\sin\theta)$ inside the unit disk is a special "average of the values $U(t)$ on the boundary." In the integrand the function $U(t)$ is multiplied by the Poisson kernel $P(r, t-\theta) = \dfrac{(1-r^2)}{1+r^2-2r\cos(t-\theta)}$ which includes the variables r and θ.

Theorem 12.8 (Poisson Integral Formula in the unit disk). Let $U(\theta)$ be the boundary values on the unit circle $C_1(0) = \{z : |z| = 1\}$,

(12.10.2) $u(\cos\theta, \sin\theta) = U(\theta)$ for $-\pi \le \theta \le \pi$. If $U(\theta)$ is both piecewise continuous and bounded, then

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,U(t)}{1+r^2-2r\cos(t-\theta)}\,dt,$$

for $0 \le r < 1, -\pi \le \theta \le \pi$, solves the Dirichlet problem and is a harmonic function in the unit disk $D_1(0) = \{z : |z| < 1\}$. Observation. The integrand is taken over the unit circle where the parameter of integration is t, and the numerator includes the function $U(t)$.

Corollary 1 (N-Value Dirichlet solution for the unit disk). Assume that $-\pi < \theta_1 < \theta_2 \ldots < \theta_N < \pi$. If $U(\theta) = u(\cos\theta, \sin\theta)$ is the boundary value function on the unit circle $C_1(0) = \{z : |z| = 1\}$,

$U(\theta) = u(\cos\theta, \sin\theta) = a_1$ for $-\pi < \theta < \theta_1$,

$U(\theta) = u(\cos\theta, \sin\theta) = a_k$ for $\theta_{k-1} < \theta < \theta_k$,

$U(\theta) = u(\cos\theta, \sin\theta) = a_N$ for $u_{N-1} < \theta < \pi$, then

$$u(r\cos\theta, r\sin\theta) = \frac{1}{\pi}\sum_{k=1}^{N} a_k \left(\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta_k-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta_{k-1}-\theta}{2}\right)\right)\right)$$

, solves the Dirichlet problem and is a harmonic function in the unit disk $D_1(0) = \{z : |z| < 1\}$.

Proof. Use the indefinite integral

$$\frac{1}{2\pi}\int \frac{(1-r^2)}{1+r^2-2r\cos(t-\theta)}\,dt = \frac{1}{\pi}\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan\left(\frac{t-\theta}{2}\right)\right).$$

Then

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-r^2)\,U(t)}{1+r^2-2r\cos(t-\theta)}\,dt$$

$$= \frac{1}{2\pi}\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}\frac{(1-r^2)\,a_k}{1+r^2-2r\cos(t-\theta)}\,dt$$

$$= \frac{1}{2\pi}\sum_{k=1}^{N}\left(2a_k\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{t-\theta}{2}\right)\right)\Bigg|_{t=\theta_{k-1}}^{t=\theta_k}\right)$$

$$= \frac{1}{\pi}\sum_{k=1}^{N} a_k \left(\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta_k-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta_{k-1}-\theta}{2}\right)\right)\right)$$

Remark. When applying formula it is necessary to pay attention and to use appropriate branches and to beware of branch cuts.

Extra Example 1. Find the function $u(x, y) = u(r\cos\theta, r\sin\theta)$ that is harmonic in the unit disk $D_1(0) = \{z : |z| < 1\}$, and takes on the boundary values

$$u(\cos\theta, \sin\theta) = U(\theta) = \begin{cases} 1, & \text{for } \frac{\pi}{2} < t < \pi, \\ 0, & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ -1, & \text{for } -\pi < t < -\frac{\pi}{2}. \end{cases}$$

Notes

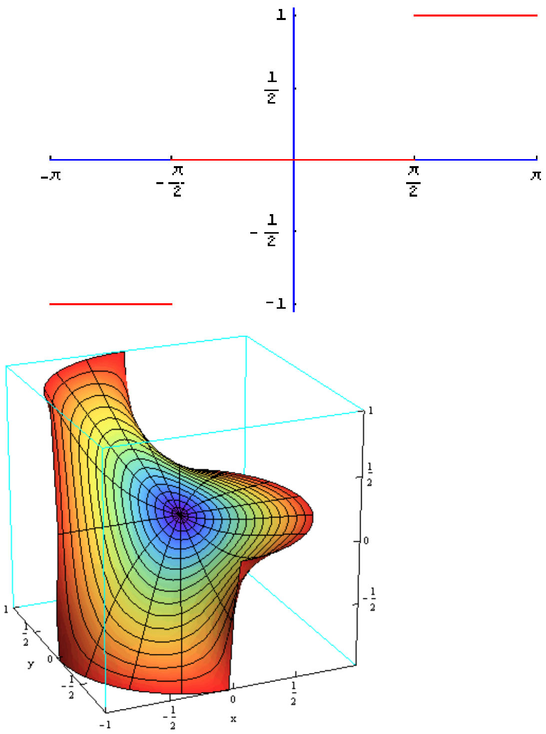


Figure 1. The graphs of $U(\theta) = u(\cos\theta, \sin\theta)$ and $u(x, y) = u(r\cos\theta, r\sin\theta)$.

Solution using Fourier series. we showed that the Fourier series for $U(\theta)$ can be written as

$$u(\cos\theta, \sin\theta) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n}\sin(n\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(i)^n + (-i)^n - 2(-1)^n}{n}\sin(n\theta)$$

.

to construct the extended Fourier series solution of the Dirichlet problem. Therefore, the Fourier series solution is

$$u(r\cos\theta, r\sin\theta) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n} r^n \sin(n\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(i)^n + (-i)^n - 2(-1)^n}{n} r^n \sin(n\theta)$$

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 State Dirichlet Problem.

Solution :

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…………………………………………………………………………………

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Q. 2 Define Extended Fourier in The Unit Disk.

Solution :

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4.3 Solution using Poisson's integral.

Use the known anti-derivative $\displaystyle\int \frac{(1-r^2)}{1+r^2-2r\cos(t-\theta)}\,dt = 2\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{t-\theta}{2}\right)\right)$ and get

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-r^2)\,U(t)}{1+r^2-2r\cos(t-\theta)}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\frac{\pi}{2}}\frac{(1-r^2)(-1)}{1+r^2-2r\,\mathrm{Cos}[t-\theta]}\,dt + \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{(1-r^2)(+1)}{1+r^2-2r\,\mathrm{Cos}[t-\theta]}\,dt$$

$$= \left(\frac{-1}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{t-\theta}{2}\right)\right)\right)\Bigg|_{t=-\pi}^{t=-\frac{\pi}{2}} + \left(\frac{1}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{t-\theta}{2}\right)\right)\right)\Bigg|_{t=\frac{\pi}{2}}^{t=\pi}$$

$$= \left(\frac{-1}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\frac{\pi}{2}-\theta}{2}\right)\right)\right) - \left(\frac{-1}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\pi-\theta}{2}\right)\right)\right)$$
$$+ \left(\frac{1}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\pi-\theta}{2}\right)\right)\right) - \left(\frac{1}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right)\right)$$

$$= \frac{\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\pi-\theta}{2}\right)\right)}{\pi} - \frac{\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\frac{\pi}{2}-\theta}{2}\right)\right)}{\pi} - \frac{\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right)}{\pi} + \frac{\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\pi-\theta}{2}\right)\right)}{\pi}$$

$$= \frac{-\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta+\pi}{2}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta+\pi}{4}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta-\pi}{4}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta-\pi}{2}\right)\right)}{\pi}$$

Therefore, the Poisson integral solution is

$$u(r\cos\theta, r\sin\theta) = \frac{-\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta+\pi}{2}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta+\pi}{4}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta-\pi}{4}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta-\pi}{2}\right)\right)}{\pi}$$

4.3.1 Solution using N-Value Dirichlet formula.

Set $a_1 = -1,\ a_2 = 0,\ a_3 = 1$ and $\theta_0 = -\pi,\ \theta_1 = -\frac{\pi}{2},\ \theta_2 = \frac{\pi}{2},\ \theta_3 = \pi$ and then use the formula and get

Notes

$$u(r\cos\theta, r\sin\theta) = \frac{1}{\pi}\sum_{k=1}^{3} a_k \left(\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta_k-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta_{k-1}-\theta}{2}\right)\right)\right)$$

$$= \frac{-1}{\pi}\left(\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\frac{\pi}{2}-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\pi-\theta}{2}\right)\right)\right)$$

$$+ \frac{0}{\pi}\left(\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{-\frac{\pi}{2}-\theta}{2}\right)\right)\right)$$

$$+ \frac{1}{\pi}\left(\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\pi-\theta}{2}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right)\right)$$

$$= \frac{\arctan\left(\frac{(1+r)\tan\left(\frac{1}{2}(-\pi-\theta)\right)}{1-r}\right) - \arctan\left(\frac{(1+r)\tan\left(\frac{1}{2}\left(-\frac{\pi}{2}-\theta\right)\right)}{1-r}\right) - \arctan\left(\frac{(1+r)\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)}{1-r}\right) + \arctan\left(\frac{(1+r)\tan\left(\frac{\pi-\theta}{2}\right)}{1-r}\right)}{\pi}$$

$$= \frac{-\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta+\pi}{2}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta+\pi}{4}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta-\pi}{4}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta-\pi}{2}\right)\right)}{\pi}$$

Therefore, the N-Value Dirichlet solution is

$$u(r\cos\theta, r\sin\theta) = \frac{-\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta+\pi}{2}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta+\pi}{4}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{2\theta-\pi}{4}\right)\right) - \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta-\pi}{2}\right)\right)}{\pi}$$

Remark. This is just a straightforward way of calculating the Poisson integral solution, and is recommended for working some of the exercises.

Extra Example 2. Find the function $v(x, y) = v(r\cos\theta, r\sin\theta)$ that is harmonic in the unit disk $D_1(0) = \{z : |z| < 1\}$, and takes on the boundary values $v(\cos\theta, \sin\theta) = V(\theta) = \theta^2$, for $-\pi < t < \pi$.

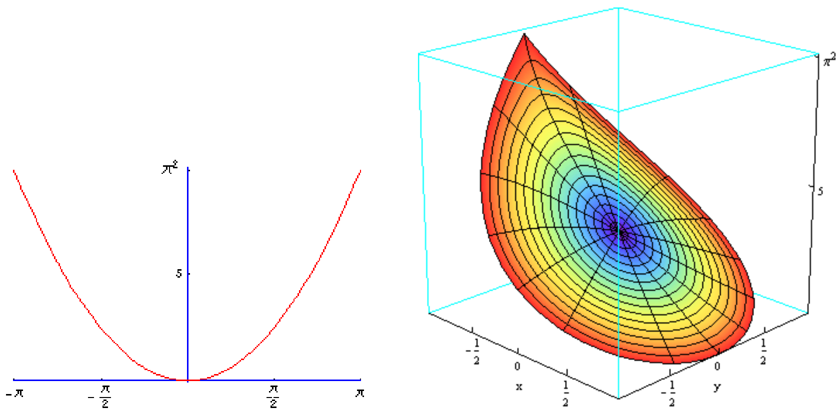


Figure 2. The graphs of $V(\theta) = v(\cos\theta, \sin\theta)$ and $v(x, y) = v(r\cos\theta, r\sin\theta)$.

where we showed that the Fourier series for $V(\theta)$ can be written as

$$v(\cos\theta, \sin\theta) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos(n\pi)}{n^2}\cos(n\theta) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(n\theta)$$

. for the extended Fourier series solution of the Dirichlet problem, and get

$$v(r\cos\theta, r\sin\theta) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos(n\pi)}{n^2} r^n \cos(nt) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} r^n \cos(nt)$$

. **Remarks.** Some integrals of the form $\int \frac{(1-r^2)U(t)}{1+r^2-2r\cos(t-\theta)}\,dt$ are readily available. For $U(t) = 1, t, t^2$ the computer algebra systems *Mathematica* and Maple can be used to obtain the following formulae.

$$\int \frac{(1-r^2)}{1+r^2-2r\,\mathrm{Cos}[t-\theta]}\,dt = 2\,\mathrm{ArcTan}\left[\frac{1+r}{1-r}\,\mathrm{Tan}\left[\frac{t-\theta}{2}\right]\right]$$

$$\int \frac{(1-r^2)\,t}{1+r^2-2r\,\mathrm{Cos}[t-\theta]}\,dt = -i\left((t-\theta)\,\mathrm{Log}\left[1-\frac{e^{i(t-\theta)}}{r}\right] - (t-\theta)\,\mathrm{Log}\left[1-e^{i(t-\theta)}r\right] + \theta\,\mathrm{Log}\left[-e^{i(t-\theta)}+r\right]\right.$$
$$\left. - \theta\,\mathrm{Log}\left[-1+e^{i(t-\theta)}r\right] - i\,\mathrm{PolyLog}\left[2, \frac{e^{i(t-\theta)}}{r}\right] + i\,\mathrm{PolyLog}\left[2, e^{i(t-\theta)}r\right]\right)$$

$$\int \frac{(1-r^2)\,t^2}{1+r^2-2r\,\mathrm{Cos}[t-\theta]}\,dt = -i\left(t^2\,\mathrm{Log}\left[1-\frac{e^{i(t-\theta)}}{r}\right] - t^2\,\mathrm{Log}\left[1-e^{i(t-\theta)}r\right] + 2\,i\,t\,\mathrm{PolyLog}\left[2, e^{i(t-\theta)}r\right]\right.$$
$$\left. - 2\,i\,t\,\mathrm{PolyLog}\left[2, \frac{e^{i(t-\theta)}}{r}\right] + 2\,\mathrm{PolyLog}\left[3, \frac{e^{i(t-\theta)}}{r}\right] - 2\,\mathrm{PolyLog}\left[3, e^{i(t-\theta)}r\right]\right)$$

The software Maple will also compute the above integrals, but the Maple syntax will use the notation **arctan, ln, polylog** and **tan**. The first integral $\int \frac{(1-r^2)}{1+r^2-2r\cos(t-\theta)}\,dt = \frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{1+r}{1-r}\tan\left(\frac{t-\theta}{2}\right)\right)$ can easily be verified using techniques from calculus.

Caveat. It is beyond the scope of this book to use the other two integrals involving **polylog**.

In this chapter we show how Fourier Series, the Fourier Transform, and the Laplace Transform are related to the study of complex analysis. First, we will introduce the Fourier series for a real-valued function $U(t)$ of the real variable t. Then we discuss Fourier transforms. Finally, we develop the Laplace transform and the complex variable techniques for finding its inverse. Our goal is to apply these ideas to solving problems, so many of the theorems are stated without proof.

Let $U(t)$ be a real-valued function that is periodic with period 2π, that is $U(t+2\pi) = U(t)$ for all t.

Notes

One such function

is $s = U(t) = \sin\left(t - \frac{\pi}{2}\right) + \frac{7}{10}\cos\left(2t - \pi - \frac{1}{4}\right) + \frac{17}{10}$. Its graph is obtained by repeating the portion of the graph in any interval of length 2π, as shown in Figure 12.1.

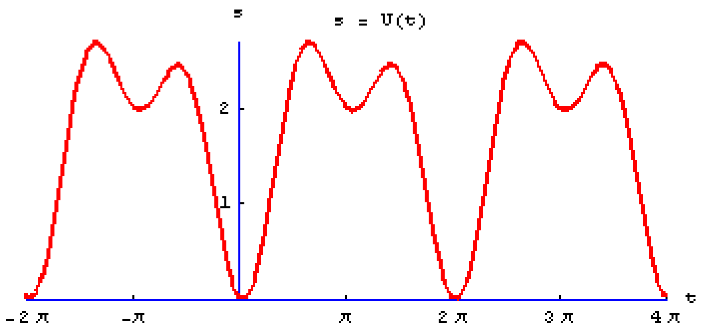


Figure 3. A function $U(t)$ with period 2π.

Familiar examples of real functions that have period 2π, are $\cos(nt)$ and $\sin(nt)$, where n is an integer. These examples raise the question of whether any periodic function can be represented by a sum of terms involving $a_n \cos(nt)$ and $b_n \sin(nt)$, where a_n and b_n are real constants. As we soon demonstrate, the answer to this question is often yes.

4.3.2 Piecewise Continuous

Definition 12.1 (Piecewise Continuous). The function $U(t)$ is piecewise continuous on the closed interval $[a, b]$, if there exists values $t_0, t_1, \ldots, t_n$ with $a = t_0 < t_1 < \ldots < t_n = b$ such that $U(t)$ is continuous in each of the open intervals $t_{k-1} < t < t_k$, for $k = 1, 2, \ldots, n$ and has left-hand and right-hand limits at each of the values t_k , for $k = 0, 1, 2, \ldots, n$.

We use the symbols $U(c^-)$ and $U(c^+)$ for the left-hand and right-hand limit, respectively, of a function $U(c)$ as t approaches the point c. The graph of a piecewise continuous function is illustrated in Figure 12.2

below, where the function $U(t)$ is

$$U(t) = \begin{cases} \frac{2}{3}\left(t - \frac{1}{2}\right)^2 + \frac{1}{4} & \text{, when } 1 \le t < 2, \\ \frac{5}{2} - (t-2)^2 & \text{, when } 2 < t < 3, \\ 1 + \frac{t-3}{4} & \text{, when } 3 < t < 4, \\ \frac{6}{5} - (t-5)^3 & \text{, when } 4 < t \le 6. \end{cases}$$

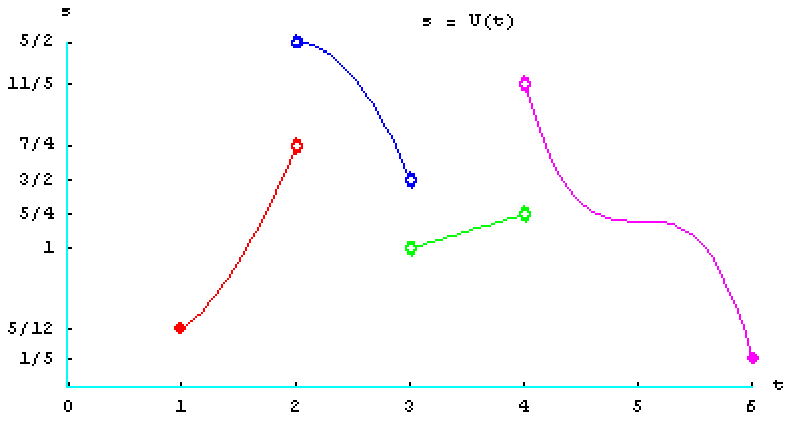


Figure 4. A piecewise continuous function $U(t)$ over the interval $[1, 6]$.

The left-hand and right-hand limits at $t_1 = 2$, $t_2 = 3$, and $t_3 = 4$ are easily determined:

At $t_1 = 2$, the left-hand limit is $U(2^-) = \lim_{t \to 2^-}\left[\frac{2}{3}\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}\right] = \frac{2}{3}\left(2 - \frac{1}{2}\right)^2 + \frac{1}{4} = \frac{7}{4}$, and the right-hand limit is $U(2^+) = \lim_{t \to 2^+}\left[\frac{5}{2} - (t-2)^2\right] = \frac{5}{2} - (2-2)^2 = \frac{5}{2}$.

At $t_2 = 3$, the left-hand limit is $U(3^-) = \lim_{t \to 3^-}\left[\frac{5}{2} - (t-2)^2\right] = \frac{5}{2} - (3-2)^2 = \frac{3}{2}$ and the right-hand limit is $U(3^+) = \lim_{t \to 3^+}\left[1 + \frac{t-3}{4}\right] = 1 + \frac{3-3}{4} = 1$.

At $t_3 = 4$, the left-hand limit is $U(4^-) = \lim_{t \to 4^-}\left[1 + \frac{t-3}{4}\right] = 1 + \frac{4-3}{4} = \frac{5}{4}$, and the right-hand limit is $U(4^+) = \lim_{t \to 4^+}\left[\frac{6}{5} - (t-5)^3\right] = \frac{6}{5} - (4-5)^3 = \frac{11}{5}$.

Notes

Definition (Fourier Series). If $U(t)$ is periodic with period 2π and is piecewise continuous on $[-\pi, \pi]$, then the Fourier Series $S(t)$ for $U(t)$ is $S(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n t) + b_n \sin(n t))$, where the coefficients a_n and b_n are given by the so-called **Euler's formulae**:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t) \cos(n t)\, dt \quad \text{for} \quad n = 0, 1, \ldots$$, and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t) \sin(n t)\, dt \quad \text{for} \quad n = 1, 2, \ldots$$.

We introduced the factor $\frac{1}{2}$ in the constant term $\frac{a_0}{2}$ on the right side of Equation (1.1) for convenience, so that we can obtain a_0 from the general formula in Equation (111) by setting $n = 0$. We explain the reasons for this strategy shortly. Theorem 1.1 deals with convergence of the Fourier series.

Theorem (Fourier Expansion). Assume that $S(t)$ is the Fourier Series for $U(t)$. If $U(t)$ and $U'(t)$ are piecewise continuous on $[-\pi, \pi]$, then $S(t)$ is convergent for all $t \in [-\pi, \pi]$. The relation $U(t) = S(t)$ holds for all $t \in [-\pi, \pi]$ where $U(t)$ is continuous. If $t = c$ is a point of discontinuity of $U(t)$, then $S(c) = \frac{U(c^-) + U(c^+)}{2}$, where $U(c^-)$ and $U(c^+)$ denote the left-hand and right-hand limits, respectively. With this understanding, we have the Fourier Series expansion:

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n t) + b_n \sin(n t))$$.

Example. The function $U(t) = \frac{t}{2}$ for $-\pi < t < \pi$, extended periodically by the equation $U(t + 2\pi) = U(t)$, has the Fourier series expansion

$$U(t) = \frac{t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(n t)$$.

Solution.

and integrating by parts, we obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\cos(n t)\,dt = \left(\frac{t\sin(n t)}{2\pi n} + \frac{\cos(n t)}{2\pi n^2}\right)\Bigg|_{t=-\pi}^{t=\pi} = 0$$

, for $n = 1, 2, \ldots$. The coefficient a_0 is obtained with the separate computation

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\,dt = \left(\frac{t^2}{4\pi}\right)\Bigg|_{t=-\pi}^{t=\pi} = 0$$

. we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\sin(n t)\,dt = \left(-\frac{t\cos(n t)}{2\pi n} + \frac{\sin(n t)}{2\pi n^2}\right)\Bigg|_{t=-\pi}^{t=\pi} = -\frac{\cos(n\pi)}{n} = \frac{(-1)^{n-1}}{n}$$

, for $n = 1, 2, \ldots$.

Substituting the coefficients $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ into Equation (12.1) produces the required solution

$$U(t) = \frac{t}{2} = \sum_{n=1}^{\infty} \frac{-\cos(n\pi)}{n}\sin(n t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(n t)$$

. The graphs of $U(t) = \frac{t}{2}$ and the first three partial sums $S_1(t) = \sin(t)$, $S_2(t) = \sin(t) - \frac{1}{2}\sin(2t)$, and $S_3(t) = \sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t)$ are shown in Figure 12.3.

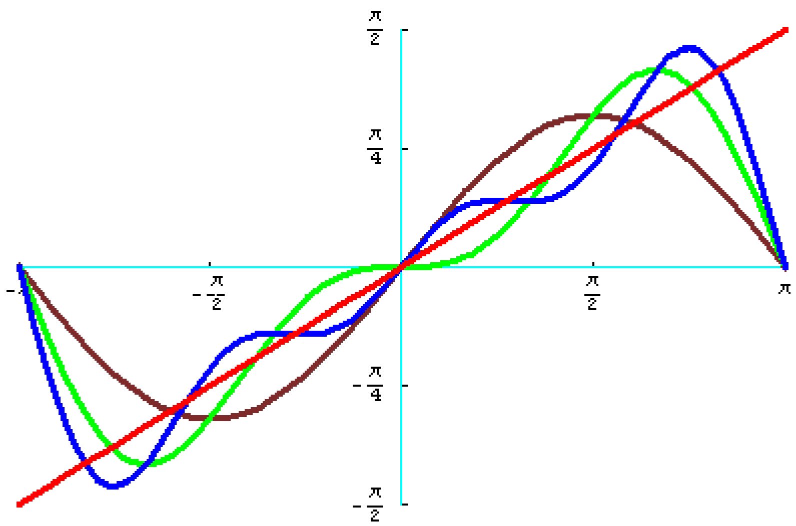


Figure. The function $U(t) = \frac{t}{2}$, and the approximations $S_1(t)$, $S_2(t)$, and $S_3(t)$.

Theorem. If $U(t)$ and $V(t)$ have Fourier series representations, then their sum $W(t) = U(t) + V(t)$ has a Fourier series representation, and the Fourier coefficients of $W(t)$ are obtained by adding the corresponding coefficients of $U(t)$ and $V(t)$.

4.3 Fourier Cosine Series

Theorem (Fourier Cosine Series). Assume that $U(t)$ is an even function and has period 2π. Here the Fourier series for $U(t)$ involves only the cosine terms, $(\text{i.e. } b_n = 0 \text{ for all } n)$, and we write

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where $$a_n = \frac{2}{\pi}\int_0^{\pi} U(t)\cos(nt)\,dt \quad \text{for } n = 0, 1, \ldots.$$

Proof.

Theorem (Fourier Sine Series). Assume that $U(t)$ is an odd function and has period 2π. Here the Fourier series for $U(t)$ involves only the sine terms, $(\text{i.e. } a_n = 0 \text{ for all } n)$, and we write

$$U(t) = \sum_{n=1}^{\infty} b_n \sin(nt),$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} U(t)\sin(nt)\,dt \quad \text{for } n = 1, 2, \ldots.$$

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 Define Fourier Cosine Series.

Solution :

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

Q. 2 State Fourier Expansion.

Solution :

……………………………………………………………………………

……………………………………………………………………………

4.5 TERMWISE INTEGRATION

Theorem (Termwise Integration). Assume that $U(t)$ has the Fourier series representation $$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Then the integral of $U(t)$ has a Fourier series representation which can be obtained by termwise integration of the Fourier series of $U(t)$, that is

$$\int_0^t U(\tau)\, d\tau = \sum_{n=1}^{\infty} \left(\frac{(a_n + (-1)^{n+1} a_0)}{n} \sin(nt) - \frac{b_n}{n} \cos(nt) \right)$$, where

we have used the expansion $$\frac{a_0}{2} t = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a_0}{n} \sin(nt).$$

Theorem 12.6 (Termwise Differentiation). Assume that both $U(t)$ and $U'(t)$ have Fourier series representation and that

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$. Then $U'(t)$ can be obtained by termwise differentiation of $U(t)$, that is

$$U'(t) = \sum_{n=1}^{\infty} (n b_n \cos(nt) - n a_n \sin(nt)).$$

Example The function $U(t) = |t|$ for $-\pi < t < \pi$, extended periodically by the equation $U(t+2\pi) = U(t)$, has the Fourier series expansion

$$U(t) = |t| = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi) - 1}{n^2} \cos(nt) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(nt)$$

, which can be written in the alternative form

$$U(t) = |t| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} \cos((2j-1)t).$$

Solution.

The function $U(t)$ is an even function; hence we can use Theorem 11.3 to conclude that $b_n = 0$ for all n and that

$$a_n = \frac{2}{\pi} \int_0^{\pi} t \cos(nt)\, dt = \left(\frac{2\cos(nt)}{\pi n^2} + \frac{2t\sin(nt)}{\pi n} \right) \Bigg|_{t=0}^{t=\pi} = \frac{2}{\pi} \frac{\cos(n\pi) - 1}{n^2} = \frac{2}{\pi} \frac{(-1)^n - 1}{n^2}$$

, for $n = 1, 2, \ldots$. The coefficient a_0 is obtained with the separate computation $$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,dt = \left(\frac{t^2}{\pi}\right)\Bigg|_{t=0}^{t=\pi} = \pi$$.

Using the $\{a_n\}_{n=0}^{\infty}$ and produces the required solution. Therefore, we have the found the Fourier series expansion

$$U(t) = |t| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi) - 1}{n^2}\cos(nt) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nt)$$

. It is easy to see that $a_{2j} = \frac{2}{\pi}\frac{(-1)^{2j} - 1}{(2j)^2} = 0$ for all $j \geq 1$, and we can express a_{2j-1} in the form

$$a_{2j-1} = \frac{2}{\pi}\frac{(-1)^{2j-1} - 1}{(2j-1)^2} = \frac{2}{\pi}\frac{(-1)^{2j-1} - 1}{(2j-1)^2} = \frac{2}{\pi}\frac{(-1)^{-1} - 1}{(2j-1)^2} = \frac{2}{\pi}\frac{-1-1}{(2j-1)^2} = \frac{-4}{\pi}\frac{1}{(2j-1)^2}$$

. Therefore, $$U(t) = |t| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\cos((2j-1)t)$$.

The graphs of $U(t) = |t|$ and the first two partial sums $S_1(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos(t)$, and $S_3(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos(t) - \frac{4}{9\pi}\cos(3t)$ are shown below.

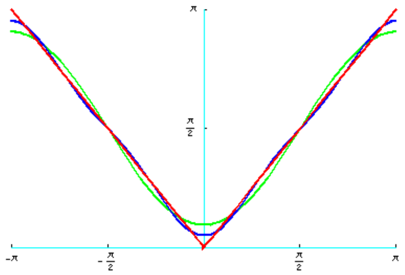


Figure.a. The function $U(t) = |t|$, and the approximations $S_1(t)$, and $S_3(t)$.

ind the Fourier Series $$S(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt))$$, by computing the coefficients with Euler's formulae:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos(nt)\,dt \quad \text{for } n = 0, 1, \ldots$$, and (

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin(nt)\,dt \quad \text{for } n = 1, 2, \ldots$$.

First, calculate $\{a_n\}_{n=0}^{\infty}$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-t)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(t)\,dt$$

$$= \left(\frac{1}{\pi}\,\frac{-t^2}{2}\right)\Bigg|_{t=-\pi}^{t=0} + \left(\frac{1}{\pi}\,\frac{t^2}{2}\right)\Bigg|_{t=0}^{t=\pi}$$

$$= \left(\frac{1}{\pi}(-0) - \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right)\right) + \left(\frac{1}{\pi}\left(\frac{\pi^2}{2}\right) - \frac{1}{\pi}(0)\right)$$

$$= \pi$$

Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos(n t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-t)\cos(n t)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(t)\cos(n t)\,dt$$

$$= \left(\frac{1}{\pi}\left(\frac{-\cos(n t) - n t\sin(n t)}{n^2}\right)\right)\Bigg|_{t=-\pi}^{t=0} + \left(\frac{1}{\pi}\left(\frac{\cos(n t) + n t\sin(n t)}{n^2}\right)\right)\Bigg|_{t=0}^{t=\pi}$$

$$= \left(\frac{1}{\pi}\left(\frac{-\cos(n0) - n0\sin(n0)}{n^2}\right) - \frac{1}{\pi}\left(\frac{-\cos(-n\pi) + n\pi\sin(-n\pi)}{n^2}\right)\right) + \left(\frac{1}{\pi}\left(\frac{\cos(n\pi) + n\pi\sin(n\pi)}{n^2}\right) - \frac{1}{\pi}\left(\frac{\cos(n0) + n0\sin(n0)}{n^2}\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{-1-0}{n^2}\right) - \frac{1}{\pi}\left(\frac{-\cos(-n\pi)+0}{n^2}\right) + \frac{1}{\pi}\left(\frac{\cos(n\pi)+0}{n^2}\right) - \frac{1}{\pi}\left(\frac{1+0}{n^2}\right)$$

$$= \frac{2}{\pi}\,\frac{\cos(n\pi) - 1}{n^2} \quad \text{for} \quad n \geq 1$$

Second, calculate $\{b_n\}_{n=1}^{\infty}$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin(n t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-t)\sin(n t)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(t)\sin(n t)\,dt$$

$$= \left(\frac{1}{\pi}\left(\frac{n t\cos(n t) - \sin(n t)}{n^2}\right)\right)\Bigg|_{t=-\pi}^{t=0} + \left(\frac{1}{\pi}\left(\frac{\sin(n t) - n t\cos(n t)}{n^2}\right)\right)\Bigg|_{t=0}^{t=\pi}$$

$$= \left(\frac{1}{\pi}\left(\frac{n0\cos(n0) - \sin(n0)}{n^2}\right) - \frac{1}{\pi}\left(\frac{-n\pi\cos(n\pi) - \sin(n\pi)}{n^2}\right)\right) + \left(\frac{1}{\pi}\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2}\right) - \frac{1}{\pi}\left(\frac{\sin(n0) - n0\cos(n0)}{n^2}\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{0-0}{n^2}\right) - \frac{1}{\pi}\left(\frac{-n\pi\cos(n\pi) - 0}{n^2}\right) + \frac{1}{\pi}\left(\frac{0 - n\pi\cos(n\pi)}{n^2}\right) - \frac{1}{\pi}\left(\frac{0-0}{n^2}\right)$$

$$= 0 \quad \text{for} \quad n \geq 1$$

Alternately, $U(t) = |t|$ is an even function so that $U(t)\sin(n t)$ is an odd function, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin(n t)\,dt = 0$ for all $n \geq 1$

. Then shows that $U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$, where the coefficients can be computed with the special formula $a_n = \frac{2}{\pi}\int_0^{\pi} U(t)\cos(nt)\,dt \quad \text{for} \quad n = 0, 1, \ldots$.

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} U(t)\,dt \\
&= \frac{2}{\pi}\int_0^{\pi} (t)\,dt \\
&= \left(\frac{2}{\pi}\,\frac{t^2}{2}\right)\Bigg|_{t=0}^{t=\pi} \\
&= \left(\frac{2}{\pi}\left(\frac{\pi^2}{2}\right) - \frac{2}{\pi}\,(0)\right) \\
&= \pi
\end{aligned}
$$

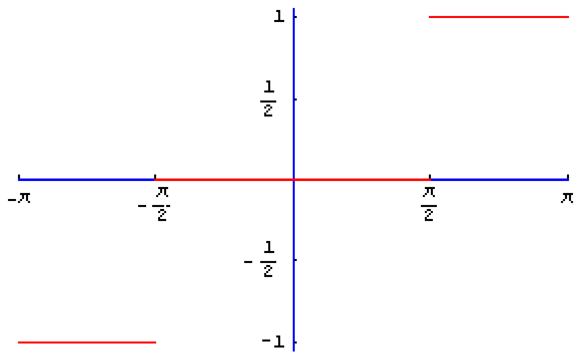
Now calculate

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} U(t)\cos(nt)\,dt \\
&= \frac{2}{\pi}\int_0^{\pi} (t)\cos(nt)\,dt \\
&= \left(\frac{2}{\pi}\left(\frac{\cos(nt) + nt\sin(nt)}{n^2}\right)\right)\Bigg|_{t=0}^{t=\pi} \\
&= \left(\frac{2}{\pi}\left(\frac{\cos(n\pi) + n\pi\sin(n\pi)}{n^2}\right) - \frac{2}{\pi}\left(\frac{\cos(n0) + n0\sin(n0)}{n^2}\right)\right) \\
&= \frac{2}{\pi}\left(\frac{\cos(n\pi) + 0}{n^2}\right) - \frac{2}{\pi}\left(\frac{1+0}{n^2}\right) \\
&= \frac{2}{\pi}\,\frac{\cos(n\pi) - 1}{n^2} \quad \text{for} \quad n \ge 1
\end{aligned}
$$

Get The Answer.

Therefore, $U(t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(n\pi) - 1}{n^2}\cos(nt)$.

Extra Example 1. Given $U(t) = \begin{cases} 1, & \text{for } \frac{\pi}{2} < t < \pi, \\ 0, & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ -1, & \text{for } -\pi < t < -\frac{\pi}{2}, \end{cases}$ extended periodically by the equation $U(t + 2\pi) = U(t)$, find the Fourier series expansion.



Find the Fourier Series $S(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n t) + b_n \sin(n t))$, by computing the coefficients with Euler's formulae:

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t) \cos(n t) \, dt \quad \text{for } n = 0, 1, \ldots$, and

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t) \sin(n t) \, dt \quad \text{for } n = 1, 2, \ldots$.

First, calculate $\{a_n\}_{n=0}^{\infty}$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t) \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{-\frac{\pi}{2}} (-1) \, dt + \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0) \, dt + \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} (1) \, dt$$

$$= \left(\frac{1}{\pi} (-t) \right) \Bigg|_{t=-\pi}^{t=-\frac{\pi}{2}} + \left(\frac{1}{\pi} (t) \right) \Bigg|_{t=\frac{\pi}{2}}^{t=\pi}$$

$$= \left(\frac{1}{\pi} \left(\frac{\pi}{2} \right) - \frac{1}{\pi} (\pi) \right) + \left(\frac{1}{\pi} (\pi) - \frac{1}{\pi} \left(\frac{\pi}{2} \right) \right)$$

$$= \frac{1}{2} - 1 + 1 - \frac{1}{2}$$

$$= 0$$

Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos(n\,t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} (-\cos(n\,t))\,dt + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0\cos(n\,t))\,dt + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos(n\,t)\,dt$$

$$= \frac{1}{\pi}\left(-\frac{\sin(n\,t)}{n}\right)\Bigg|_{t=-\pi}^{t=-\frac{\pi}{2}} + \frac{1}{\pi}\left(\frac{\sin(n\,t)}{n}\right)\Bigg|_{t=\frac{\pi}{2}}^{t=\pi}$$

$$= \frac{1}{\pi}\left(-\frac{\sin\left(-\frac{n\pi}{2}\right)}{n} + \frac{\sin(-n\,\pi)}{n}\right) + \frac{1}{\pi}\left(\frac{\sin(n\,\pi)}{n} - \frac{\sin\left(\frac{n\pi}{2}\right)}{n}\right)$$

$$= \frac{1}{\pi}\left(\frac{\sin\left(\frac{n\pi}{2}\right)}{n} - \frac{\sin(n\,\pi)}{n} + \frac{\sin(n\,\pi)}{n} - \frac{\sin\left(\frac{n\pi}{2}\right)}{n}\right)$$

$$= 0 \quad \text{for} \quad n \geq 1$$

Second, calculate $\{b_n\}_{n=1}^{\infty}$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin(n\,t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} (-\sin(n\,t))\,dt + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0\sin(n\,t))\,dt + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin(n\,t)\,dt$$

$$= \frac{1}{\pi}\left(\frac{\cos(n\,t)}{n}\right)\Bigg|_{t=-\pi}^{t=-\frac{\pi}{2}} + \frac{1}{\pi}\left(-\frac{\cos(n\,t)}{n}\right)\Bigg|_{t=\frac{\pi}{2}}^{t=\pi}$$

$$= \frac{1}{\pi}\left(\frac{\cos\left(-\frac{n\pi}{2}\right)}{n} - \frac{\cos(-n\,\pi)}{n}\right) + \frac{1}{\pi}\left(-\frac{\cos(n\,\pi)}{n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n}\right)$$

$$= \frac{1}{\pi}\left(\frac{\cos\left(\frac{n\pi}{2}\right)}{n} - \frac{\cos(n\,\pi)}{n} - \frac{\cos(n\,\pi)}{n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n}\right)$$

$$= \frac{2}{\pi}\,\frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\,\pi)}{n} \quad \text{for} \quad n \geq 1$$

Alternately, $U(t) = \begin{cases} 1, & \text{for} \quad \frac{\pi}{2} < t < \pi, \\ 0, & \text{for} \quad -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ -1, & \text{for} \quad -\pi < t < -\frac{\pi}{2}. \end{cases}$ is an odd function so that $U(t)\cos(n\,t)$ is an odd function, and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos(nt)\,dt = 0$ for all $n \geq 0$. shows that

$U(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$, where the coefficients can be computed with the special formula $b_n = \frac{2}{\pi}\int_0^{\pi} U(t)\sin(nt)\,dt$ for $n = 1, 2, \ldots$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} U(t)\sin(nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (0\sin(nt))\,dt + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin(nt)\,dt$$

$$= \frac{2}{\pi}\left(-\frac{\cos(nt)}{n}\right)\Bigg|_{t=\frac{\pi}{2}}^{t=\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\cos(n\pi)}{n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n}\right)$$

Now calculate

$$= \frac{2}{\pi}\,\frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n} \quad \text{for } n \geq 1$$

Get The Answer.

Therefore, $$U(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n}\sin(nt)$$.

4.5.1 Characterization Of Harmonic Functions By Mean Value Property

Let $\phi(x, y)$ be a continuous real-valued function of the two real variables x and y that is defined on a domain D. (Recall from Section 1.6 that a domain D is an connected and open set of points in the complex plane.) The partial differential equation

$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$, is known as Laplace's equation and is sometimes referred to as the potential equation.

If $\phi_x(x, y)$, $\phi_y(x, y)$, $\phi_{xx}(x, y)$, $\phi_{xy}(x, y)$, $\phi_{yx}(x, y)$ and $\phi_{yy}(x, y)$ are

Notes

all continuous, and if $\phi(x, y)$ satisfies Laplace's equation, then $\phi(x, y)$ is called a harmonic function.

In calculus we might have been asked to show that polynomial functions like $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$, and transcendental functions like $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$, and $u(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$ and $v(x, y) = \arctan\left(\frac{y}{x}\right)$, are all harmonic functions. These pairs of functions are not chosen at random, and there is an intimate relationship between them, they are called the conjugate "harmonic functions." It is our goal to understand how this concept is tied in with analytic functions.

On the practical side, harmonic functions are important in the areas of applied mathematics, engineering, and mathematical physics. Harmonic functions are used to solve problems involving steady state temperatures, two-dimensional electrostatics, and ideal fluid flow. we will show how complex analysis techniques are used to solve these problems. For example, the function

$$\phi(x, y) = \frac{1}{\pi}\arctan\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\arctan\left(\frac{y}{x+1}\right),$$

is harmonic in the upper half plane and takes on the boundary values

$$\phi(x, 0) = 1 \text{ when } |x| < 1$$

and $\phi(x, 0) = 0$ when $|x| > 1$.

4.6 THE POLAR FORM OF A COMPLEX NUMBER

The fundamental trigonometric identity (i.e the Pythagorean theorem) is

$$\cos^2\theta + \sin^2\theta = 1$$

From this we can see that the complex numbers

$$\cos\theta + i\sin\theta$$

are points on the circle of radius one centered at the origin.

Think of the point $\cos\theta + i\sin\theta$ moving counterclockwise around the circle as the real number θ moves from left to right. Similarly, the point moves clockwise if θ decreases. And whether θ increases or decreases, the point returns to the same position on the circle whenever θ changes by 2π or by 4π or by $2k\pi$ where *k* is any integer.

Exercise: Verify that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

Exercise: Prove *de Moivre's formula*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Now picture a fixed complex number on the unit circle

$$z = \cos\theta + i\sin\theta \qquad |z| = 1.$$

Consider multiples of *z* by a real, positive number *r*.

$$rz = r(\cos\theta + i\sin\theta) \qquad |rz| = r|z| = r.$$

As *r* grows from 1, our point moves out along the ray whose tail is at the origin and which passes through the point *z*. As *r* shrinks from 1 toward zero, our point moves inward along the same ray toward the origin. The modulus of the point is *r*. We call the angle θ which this ray makes with the x-axis, the *argument* of the number *z*. All the numbers *rz* have the same argument. We write

$$\arg rz = \theta$$

Just as a point in the plane is completely determined by its polar coordinates (r, θ), a complex number is completely determined by its modulus and its argument.

Notice that the argument is not defined when *r*=0 and in any case is only determined up to an integer multiple of 2π.

Why not just use polar coordinates? What's new about this way of thinking about points in the plane

4.7 SUMMARY

We study in this unit Cosine and Sine for harmonic function. We study an approximation using Partial sum.We study extended Fourier series in the unit disk. We study piecewise continuous and Fourier cosine series.

4.8 KEYWORD

Caveat : a notice, especially in a probate, that certain actions may not be taken without informing the person who gave the notice

Piecewise : a *piecewise-defined* function (also called a *piecewise* function or a hybrid function) is a function *defined* by multiple sub-functions, each sub-function applying to a certain interval of the main function's domain, a sub-domain

Argument : a reason or set of reasons given in support of an idea, action or theory

4.9 QUESTIONS FOR REVIEW

Q. 1 Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be an analytic function on a domain D. Then both $u(x,y)$ and $v(x,y)$ are harmonic functions on D. In other words, the real and imaginary parts of an analytic function are harmonic.

Proof. Since $f(z)$ is differentiable on D, the Cauchy-Riemann equations) imply that $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$

Q. 2 Show that $u(x,y) = x^2 - y^2$ is a harmonic function and find a conjugate harmonic function $v(x,y)$, and an analytic function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

Q. 3 Show that $v(x, y) = 3x^2 y - y^3$ is a harmonic conjugate of $u(x, y) = x^3 - 3xy^2$

Q. 4 Given the harmonic functions $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2 y - y^3$, and the analytic function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$.

Q. 5 Let $u(x, y)$ be harmonic in an ϵ-neighborhood of the point (x_0, y_0). Then there exists a conjugate harmonic function $v(x, y)$ defined in this neighborhood such that $f(z) = u(x, y) + iv(x, y)$, is an analytic function.

Q. 6 Show that the harmonic function $\phi(x, y) = x^2 - y^2$ is the scalar potential function for the fluid flow $\vec{V}(x, y) = 2x - i2y$.

Q. 7 Assume that $S(t)$ is the Fourier Series for $U(t)$. If $U(t)$ and $U'(t)$ are piecewise continuous on $[-\pi, \pi]$, then $S(t)$ is convergent for all $t \in [-\pi, \pi]$. The relation $U(t) = S(t)$ holds for all $t \in [-\pi, \pi]$ where $U(t)$ is continuous. If $t = c$ is a point of discontinuity of $U(t)$, then $S(c) = \dfrac{U(c^-) + U(c^+)}{2}$,
where $U(c^-)$ and $U(c^+)$ denote the left-hand and right-hand limits, respectively. With this understanding, we have the Fourier Series expansion:

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

Q. 8 The function $U(t) = \dfrac{t}{2}$ for $-\pi < t < \pi$, extended periodically by the equation $U(t + 2\pi) = U(t)$, has the Fourier series expansion

$$U(t) = \frac{t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(nt)$$

Q. 9 Assume that $U(t)$ is an even function and has period 2π. Here the Fourier series for $U(t)$ involves only the cosine terms, (i.e. $b_n = 0$ for all n), and we write $U(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$,

Q. 10 Assume that $U(t)$ has the Fourier series representation

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

. Then the integral of $U(t)$ has a Fourier series representation which can be obtained by termwise integration of the Fourier series of $U(t)$, that is

$$\int_0^t U(\tau)\, d\tau = \sum_{n=1}^{\infty} \left(\frac{(a_n + (-1)^{n+1} a_0)}{n} \sin(nt) - \frac{b_n}{n} \cos(nt) \right),$$

where we have used the expansion

$$\frac{a_0}{2} t = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a_0}{n} \sin(nt)$$

Q. 11 Assume that both $U(t)$ and $U'(t)$ have Fourier series representation and that

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Then $U'(t)$ can be obtained by termwise differentiation of $U(t)$, that is

$$U'(t) = \sum_{n=1}^{\infty} (n\, b_n \cos(nt) - n\, a_n \sin(nt)).$$

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4.11 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1

2 Check in Section 2.1

Check In Progress-II

Answer Q. 1 Check in section 4.3

Q 2 Check in Section 4.2

UNIT 5 : GEOMETRIC SERIES AND CONVERGENCE

STRUCTURE

5.0 OBJECTIVE

- Learn about geometric series
- To know convergence test
- Test series of convergence
- Test convergences Ratio test
- Learn common ratio test

5.1 INTRODUCTION

In mathematics, a **geometric series** is a series with a constant ratio between successive terms.

Notes

½ + 1/3 + ¼ + 1/5 + 1/6 ……………………………..

is geometric, because each successive term can be obtained by multiplying the previous term by 1/2.

Geometric series are among the simplest examples of infinite series with finite sums, although not all of them have this property. Historically, geometric series played an important role in the early development of calculus, and they continue to be central in the study of convergence of series. Geometric series are used throughout mathematics, and they have important applications in physics, engineering, biology, economics, computer science, queueing theory, and finance.

5.1.1 Common Ratio

The terms of a geometric series form a geometric progression, meaning that the ratio of successive terms in the series is constant. This relationship allows for the representation of a geometric series using only two terms, *r* and *a*. The term *r* is the common ratio, and *a* is the first term of the series. As an example, the geometric series is given in the introduction,

½ + ¼ + 1/8 + 1/16 + …………………..

May simply be written as

a + ar^2 + ar^3 + ar^4 + ar^5 + ………………………….

With a = ½ and r = ½

The behavior of the terms depends on the common ratio *r*:

- If *r* is between −1 and +1, the terms of the series approach zero in the limit (becoming smaller and smaller in magnitude), and the series converges to a sum. In the case above, where *r* is 1/2, the series converges to 1.
- If *r* is **greater than one** or **less than minus one** the terms of the series become larger and larger in magnitude. The sum of the terms also gets larger and larger, and the series has no sum. (The series diverges.)

- If r is **equal to one**, all of the terms of the series are the same. The series diverges.
- If r is **minus one** the terms take two values alternately (e.g. 2, −2, 2, −2, 2,...). The sum of the terms oscillates between two values (e.g. 2, 0, 2, 0, 2,...). This is a different type of divergence and again the series has no sum. See for example Grandi's series: $1 - 1 + 1 - 1 + \cdots$.

Sum

The sum of a geometric series is finite as long as the absolute value of the ratio is less than 1; as the numbers near zero, they become insignificantly small, allowing a sum to be calculated despite the series containing infinitely many terms. The sum can be computed using the self-similarity of the series.

Zeno's Paradoxes

The convergence of a geometric series reveals that a sum involving an infinite number of summands can indeed be finite, and so allows one to resolve many of Zeno's paradoxes. For example, Zeno's dichotomy paradox maintains that movement is impossible, as one can divide any finite path into an infinite number of steps wherein each step is taken to be half the remaining distance. Zeno's mistake is in the assumption that the sum of an infinite number of finite steps cannot be finite. This is of course not true, as evidenced by the convergence of the geometric series with

$r = 1/2$.

5.1.2 Special Series

In this section, we are going to take a brief look at three special series. Actually, special may not be the correct term. All three have been named which makes them special in some way, however, the main reason that we're going to look at two of them in this section is that they are the only types of series that we'll be looking at for which we will be able to get actual values for the series. The third type is divergent and so won't have a value to worry about.

In general, determining the value of a series is very difficult and outside of these two kinds of series that we'll look at in this section, we will not be determining the value of series in this chapter.

So, let's get started.

Geometric Series

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} a\, r^{n-1}$$

or, with an **index shift** the geometric series will often be written as,

$$\sum_{n=0}^{\infty} a\, r^{n}$$

These are identical series and will have identical values, provided they converge of course.

If we start with the first form it can be shown that the partial sums are,

$$s_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a r^n}{1-r}$$

The series will converge provided the partial sums form a convergent sequence, so let's take the limit of the partial sums.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{a r^n}{1-r}\right) = \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{a r^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n$$

Now, from the Sequences section we know that the limit above will exist and be finite provided $-1 < r \le 1$. However, note that we can't let $r=1$ since this will give division by zero. Therefore, this will exist and be finite provided $-1 < r < 1$ and in this case the limit is zero and so we get,

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

Therefore, a geometric series will converge if $-1 < r < 1$, which is usually written $|r| < 1$, its value is,

$$n\text{-}\sum_{n=1}^{\infty} a\, r^{n-1} = \sum_{n=0}^{\infty} a\, r^{n} = \frac{a}{1-r}$$

Note that in using this formula we'll need to make sure that we are in the correct form. In other words, if the series starts at $n=0$ then the exponent on the r must be n. Likewise, if the series starts at $n=1$ then the exponent on the r must be $n-1$

5.2 CONVERGENCE AND DIVERGENCE IN SERIES

Definition of Convergence and Divergence in Series

The n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is given by $S_n = a_1 + a_2 + a_3 + ... + a_n$. If the sequence of these partial sums $\{S_n\}$ converges to L, then the sum of the series converges to L. If $\{S_n\}$ diverges, then the sum of the series diverges.

Operations on Convergent Series

If $\sum a_n = A$, and $\sum b_n = B$, then the following also converge as indicated:

$$\sum ca_n = cA \quad \sum (a_n + b_n) = A + B \quad \sum (a_n - b_n) = A - B$$

5.2.1 Listing of Convergence Tests

Absolute Convergence

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

Alternating Series Test

If for all n, a_n is positive, non-increasing (i.e. $0 < a_{n+1} <= a_n$), and approaching zero, then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ both converge. If the alternating series converges, then the remainder $R_N = S - S_N$ (where S is the exact sum of the infinite series and S_N is the sum of the first N terms of the series) is bounded by $|R_N| <= a_{N+1}$

Deleting the first N Terms

If N is a positive integer, then the series

$\sum_{n=1}^{\infty} a_n$ and $\infty \sum a_n$ n=N+1

both converge or both diverge.

Direct Comparison Test

If 0 <= a_n <= b_n for all n greater than some positive integer N, then the following rules apply: If $\sum_{n=1}^{\infty}$ b_n converges, then $\sum_{n=1}^{\infty}$ a_n converges.

If $\sum_{n=1}^{\infty}$ a_n diverges, then $\sum_{n=1}^{\infty}$ b_n diverges.

Geometric Series Convergence

The geometric series is given by $\sum_{n=0}^{\infty}$ a r^n = a + a r + a r^2 + a r^3 +... If |r| < 1 then the following geometric series converges to a / (1 - r).

If |r| >= 1 then the above geometric series diverges.

Integral Test

If for all n >= 1, f(n) = a_n, and f is positive, continuous, and decreasing then

$\sum_{n=1}^{\infty}$ a_n and $\int_1^{\infty}$ a_n

either both converge or both diverge. If the above series converges, then the remainder R_N = S - S_N (where S is the exact sum of the infinite series and S_N is the sum of the first N terms of the series) is bounded by 0< = R_N <= $\int$ (N..∞) f(x) dx.

Limit Comparison Test

If lim (n-->) (a_n / b_n) = L, where a_n, b_n > 0 and L is finite and positive, then the series $\sum_{n=1}^{\infty}$ a_n and $\sum_{n=1}^{\infty}$ b_n either both converge or both diverge.

n^{th}-Term Test for Divergence

If the sequence {a_n} does not converge to zero, then the series $\sum_{n=1}^{\infty}$ a_n diverges.

p-Series Convergence

The p-series is given by $\sum_{n=1}^{\infty} 1/n^p = 1/1^p + 1/2^p + 1/3^p + \ldots$ where $p > 0$ by definition. If $p > 1$, then the series converges. If $0 < p <= 1$ then the series diverges.

Ratio Test

If for all n, $n \neq 0$, then the following rules apply: Let $L = \lim (n \to \infty) | a_{n+1} / a_n |$. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. If $L = 1$, then the test in *inconclusive*.

Root Test

Let $L = \lim (n \to \infty) | a_n |^{1/n}$. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. If $L = 1$, then the test in *inconclusive*.

Taylor Series Convergence

If f has derivatives of all orders in an interval I centered at c, then the Taylor series converges as indicated: $\sum_{n=0}^{\infty} (1/n!) f^{(n)}(c) (x - c)^n = f(x)$ if and only if $\lim (n \to \infty) R_n = 0$ for all x in I. The remainder $R_N = S - S_N$ of the Taylor series (where S is the exact sum of the infinite series and S_N is the sum of the first N terms of the series) is equal to $(1/(n+1)!) f^{(n+1)}(z) (x - c)^{n+1}$, where z is some constant between x and c.

5.3 SEQUENCES AND SERIES

In formal terms, a complex sequence is a function whose domain is the positive integers and whose range is a subset of the complex numbers. The following are examples of sequences:

$$f(n) = \left(2 - \frac{1}{n}\right) + \left(5 + \frac{1}{n}\right) i \qquad (n = 1, 2, 3, \ldots);$$

$$g(n) = e^{i \frac{\pi n}{4}} \qquad (n = 1, 2, 3, \ldots);$$

Notes

$$h(k) = 5 + 3i + \left(\frac{1}{1+i}\right)^k \qquad (k = 1, 2, 3, \ldots);$$

$$r(n) = \left(\frac{1}{4} + \frac{i}{2}\right)^n \qquad (n = 1, 2, 3, \ldots).$$

For convenience, at times we use the term sequence rather than complex sequence. If we want a function s to represent an arbitrary sequence, we can specify it by writing $s(1) = z_1, s(2) = z_2, s(3) = z_3$, and so on. The values $z_1, z_2, z_3, \cdots$, are called the terms of a sequence, and mathematicians, being generally lazy when it comes to such things, often refer to z_1, z_2, z_3, etc. as the sequence itself, even though they are really speaking of the range of the sequence when they do so. You will usually see a sequence written as $\{z_n\}_{n=1}^{\infty}$, $\{z_n\}_1^{\infty}$, or when the indices are understood, as $\{z_n\}$. Mathematicians are also not so fussy about starting a sequence at z_1, so that $\{z_n\}_{n=-1}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$, etc., would also be acceptable notation, provided all terms were defined. For example, the sequence r given by Equation (4-4) could be written in a variety of ways: $\left\{\left(\frac{1}{4}+\frac{i}{2}\right)^n\right\}_{n=1}^{\infty}$, $\left\{\left(\frac{1}{4}+\frac{i}{2}\right)^n\right\}_1^{\infty}$, $\left\{\left(\frac{1}{4}+\frac{i}{2}\right)^n\right\}_{n=1}^{\infty}$, $\left\{\left(\frac{1}{4}+\frac{i}{2}\right)^{n+2}\right\}_{n=-2}^{\infty}$, $\left\{\left(\frac{1}{4}+\frac{i}{2}\right)^k\right\}_{k=1}^{\infty}$,...

The sequences f and g given by Equations (4-1) and (4-2) behave differently as n gets larger. The terms in Equation (4-1) approach $2 + 5i = (2, 5)$, but those in Equation (4-2) do not approach any particular number, as they oscillate around the eight eighth roots of unity on the unit circle. Informally, the sequence $\{z_n\}_1^{\infty}$ has ζ as its limit as n approaches infinity, provided the terms z_n can be made as close as we want to ζ by making n large enough. When this happens, we write $\lim_{n\to\infty} z_n = \zeta$ or $z_n \to \zeta$ as $n \to \infty$. If $\lim_{n\to\infty} z_n = \zeta$, we say that the sequence $\{z_n\}_1^{\infty}$ converges to ζ. We need a rigorous definition for Statement (4-5), however, if we are to do honest mathematics.

5.4 LIMIT OF A SEQUENCE

Definition 4.1 (Limit of a Sequence). $\lim_{n\to\infty} z_n = \zeta$ means that for any real number $\epsilon > 0$ there corresponds a positive integer N_ϵ (which depends on

ϵ) such that $z_n \in D_\epsilon(\xi)$ whenever $n > N_\epsilon$. That is $|\xi - z_n| < \epsilon$ whenever $n > N_\epsilon$. Figure 4.1 illustrates a convergent sequence.

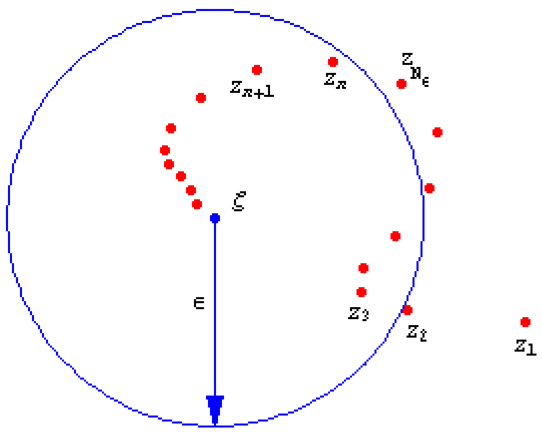


Figure 4.1 A sequence $\{z_n\}_1^\infty$ that converges to ξ. (If $n > N_\epsilon$ then $z_n \in D_\epsilon(\xi)$.)

Remark The reason we use the notation N_ϵ is to emphasize the fact that this number depends on our choice of ϵ. Sometimes it will be convenient to drop the subscript.

In form, Definition is exactly the same as the corresponding definition for limits of real sequences. In fact, a simple criterion casts the convergence of complex sequences in terms of the convergence of real sequences.

Theorem Let $z_n = x_n + i y_n$ and $\xi = u + i v$. Then $\lim_{n\to\infty} z_n = \xi$, iff $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 Define Limit of a Sequence.

Solution :

…………………………………………………………………………………

Notes

……………………………………………………………………………

……………………………………………………………………………

Q. 2 Define Geometric series.

Solution :

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

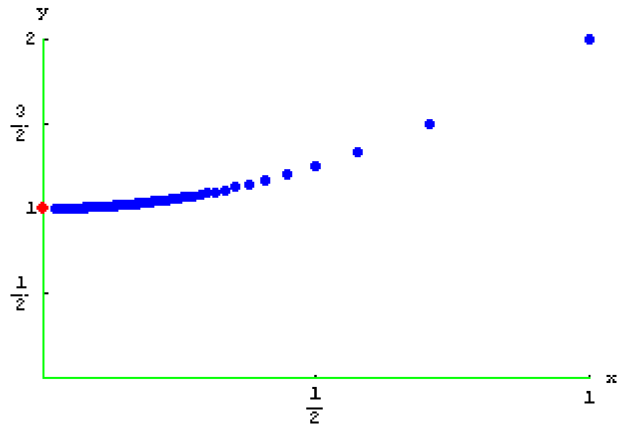
Example 4.1. Find the limit of the sequence $\{z_n\} = \left\{ \frac{\sqrt{n} + i(n+1)}{n} \right\}$.

Solution. We write $z_n = x_n + i\,y_n = \frac{1}{\sqrt{n}} + i\,\frac{n+1}{n}$. Using results concerning sequences of real numbers, we find

that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$. Therefore

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{\sqrt{n} + i(n+1)}{n} = 0 + i = i$.

Aside. Just for fun, we can graph some of the terms in this complex sequence.



The sequence of points $\{z_n\} = \left\{ \frac{\sqrt{n} + i(n+1)}{n} \right\}$ converges to $z = i$.

We write $z_n = x_n + i\,y_n = \frac{1}{\sqrt{n}} + i\,\frac{n+1}{n}$. Using results concerning sequences of real numbers, we find that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$. Therefore

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{\sqrt{n} + i\,(n+1)}{n} = 0 + i = i$$

$$z_n = \frac{1}{\sqrt{n}} + \frac{i\,(1+n)}{n}$$

$$x_n = \frac{1}{\sqrt{n}}$$

$$y_n = \frac{1+n}{n}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1+n}{n} = 1$$

$$\{z_n\} = \left\{1+2i,\ \frac{3i}{2}+\frac{1}{\sqrt{2}},\ \frac{4i}{3}+\frac{1}{\sqrt{3}},\ \frac{1}{2}+\frac{5i}{4},\ \frac{6i}{5}+\frac{1}{\sqrt{5}},\ \frac{7i}{6}+\frac{1}{\sqrt{6}},\ \frac{8i}{7}+\frac{1}{\sqrt{7}},\ \frac{9i}{8}+\frac{1}{2\sqrt{2}},\ \frac{1}{3}+\frac{10i}{9},\ \ldots\right\}$$

$$\{z_n\} = \{1.+2.\,i,\ 0.707107+1.5\,i,\ 0.57735+1.33333\,i,\ 0.5+1.25\,i,\ 0.447214+1.2\,i,\ 0.408248+1.16667\,i,\ 0.377964+1.14286\,i,\ 0.353553+1.125\,i,\ 0.333333+1.11111\,i,\ \ldots\}$$

The limit of the sequence is

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{i\,(1+n)}{n}\right)$$

$$\lim_{n\to\infty} z_n = 0 + i\,(1)$$

$$\lim_{n\to\infty} z_n = i$$

$$z_n = z(n) = \frac{1}{\sqrt{n}} + \frac{i\,(1+n)}{n}$$

Notes

z_1	=	$1+2i$	=	$1.000000000000 + 2.000000000000\,i$
z_2	=	$\frac{3i}{2}+\frac{1}{\sqrt{2}}$	=	$0.707106781187 + 1.500000000000\,i$
z_3	=	$\frac{4i}{3}+\frac{1}{\sqrt{3}}$	=	$0.577350269190 + 1.333333333330\,i$
z_4	=	$\frac{1}{2}+\frac{5i}{4}$	=	$0.500000000000 + 1.250000000000\,i$
z_5	=	$\frac{6i}{5}+\frac{1}{\sqrt{5}}$	=	$0.447213595500 + 1.200000000000\,i$
z_6	=	$\frac{7i}{6}+\frac{1}{\sqrt{6}}$	=	$0.408248290464 + 1.166666666670\,i$
z_7	=	$\frac{8i}{7}+\frac{1}{\sqrt{7}}$	=	$0.377964473009 + 1.142857142860\,i$
z_8	=	$\frac{9i}{8}+\frac{1}{2\sqrt{2}}$	=	$0.353553390593 + 1.125000000000\,i$
z_9	=	$\frac{1}{3}+\frac{10i}{9}$	=	$0.333333333333 + 1.111111111110\,i$
z_{10}	=	$\frac{11i}{10}+\frac{1}{\sqrt{10}}$	=	$0.316227766017 + 1.100000000000\,i$
z_{11}	=	$\frac{12i}{11}+\frac{1}{\sqrt{11}}$	=	$0.301511344578 + 1.090909090910\,i$
z_{12}	=	$\frac{13i}{12}+\frac{1}{2\sqrt{3}}$	=	$0.288675134595 + 1.083333333330\,i$
z_{13}	=	$\frac{14i}{13}+\frac{1}{\sqrt{13}}$	=	$0.277350098113 + 1.076923076920\,i$
z_{14}	=	$\frac{15i}{14}+\frac{1}{\sqrt{14}}$	=	$0.267261241912 + 1.071428571430\,i$
z_{15}	=	$\frac{16i}{15}+\frac{1}{\sqrt{15}}$	=	$0.258198889747 + 1.066666666670\,i$
z_{16}	=	$\frac{1}{4}+\frac{17i}{16}$	=	$0.250000000000 + 1.062500000000\,i$
z_{17}	=	$\frac{18i}{17}+\frac{1}{\sqrt{17}}$	=	$0.242535625036 + 1.058823529410\,i$
z_{18}	=	$\frac{19i}{18}+\frac{1}{3\sqrt{2}}$	=	$0.235702260396 + 1.055555555560\,i$
z_{19}	=	$\frac{20i}{19}+\frac{1}{\sqrt{19}}$	=	$0.229415733871 + 1.052631578950\,i$
z_{20}	=	$\frac{21i}{20}+\frac{1}{2\sqrt{5}}$	=	$0.223606797750 + 1.050000000000\,i$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}+\frac{i\,(1+n)}{n}\right) = 0 + i\,(1) = i$$

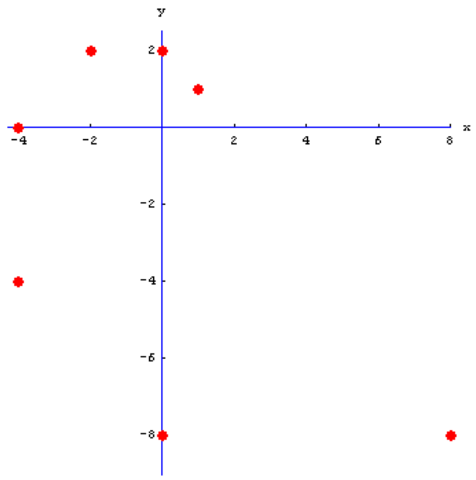
We see that the limit of the sequence $\{z_n\}=\left\{\frac{\sqrt{n}+i\,(n+1)}{n}\right\}$ is i. However, the real part is converging slowly to 0 and the imaginary part is converging a little faster to i.

Example 4.2. Show that the sequence $\{z_n\}=\{(1+i)^n\}$ diverges.

$$z_n = (1+i)^n = x_n + i\,y_n$$

Solution. We have $z_n = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} + i\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$ The real sequences $x_n = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$ and $y_n = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$ both exhibit divergent oscillations, so we conclude that $z_n = (1+i)^n$ diverges.

Aside. Just for fun, we can graph some of the terms in this divergent complex sequence.



The sequence of points $\{z_n\} = \{(1+i)^n\}$ diverges.

$$\{z_n\} = \{(1+i)^n\}$$
$$\{z_n\} = \left\{2^{n/2}\,\mathrm{Cos}\left[\frac{n\pi}{4}\right] + i\,2^{n/2}\,\mathrm{Sin}\left[\frac{n\pi}{4}\right]\right\}$$
$$\{z_n\} = \{1+i,\ 2i,\ -2+2i,\ -4,\ -4-4i,\ -8i,\ 8-8i,\ 16,\ 16+16i,\ 32i,\ \ldots\}$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (1+i)^n = \mathrm{ComplexInfinity}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2^{n/2}\,\mathrm{Cos}\left[\frac{n\pi}{4}\right] = \mathrm{Interval}[\{-\infty, \infty\}]$$
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} 2^{n/2}\,\mathrm{Sin}\left[\frac{n\pi}{4}\right] = \mathrm{Interval}[\{-\infty, \infty\}]$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (1+i)^n = \mathrm{Interval}[\{-\infty, \infty\}] + i\,(\mathrm{Interval}[\{-\infty, \infty\}])$$
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (1+i)^n = (1+i)\,\mathrm{Interval}[\{-\infty, \infty\}]$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2^{n/2}\,\mathrm{Cos}\left[\frac{n\pi}{4}\right] = \mathrm{Interval}[\{-\infty, \infty\}]$$

$\{x_n\}$ diverges

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} 2^{n/2}\,\mathrm{Sin}\left[\frac{n\pi}{4}\right] = \mathrm{Interval}[\{-\infty, \infty\}]$$

$\{y_n\}$ diverges

$$z_n = z(n) = (1 + i)^n$$
$$\{z_n\} = \{1 + i, 2i, -2 + 2i, -4, -4 - 4i, -8i, 8 - 8i, 16, 16 + 16i, 32i, \ldots\}$$

$$\lim_{n\to\infty} (1 + i)^n = \text{Interval}[\{-\infty, \infty\}] + i\ \text{Interval}[\{-\infty, \infty\}]$$

$\{z_n\}$ diverges

We see that the sequence $\{z_n\} = \{(1 + i)^n\}$ is divergent.

Definition 4.2 (Bounded Sequence). A complex sequence $\{z_n\}$ is bounded provided that there exists a positive real number R and an integer N such that $|z_n| < R$ for all $n > N$. In other words, for $n > N$, the sequence $\{z_n\}$ is contained in the disk $D_R(0)$.

Bounded sequences play an important role in some newer developments in complex analysis that are discussed in Section 4.2. A theorem from real analysis stipulates that convergent sequences are bounded. The same result holds for complex sequences.

Theorem 4.2. If $\{z_n\}$ is a convergent sequence, then $\{z_n\}$ is bounded.

As with real numbers, we also have the following definition.

Definition 4.3 (Cauchy Sequence). The sequence $\{z_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists a positive integer N_ϵ, such that if $n, m > N_\epsilon$, then $|z_n - z_m| < \epsilon$, or, equivalently, $z_n - z_m \in D_\epsilon(0)$.

The following should now come as no surprise.

4.2 Cauchy Sequences Convergence

Theorem 4.3, (Cauchy Sequences Converge). If $\{z_n\}$ is a Cauchy sequence, then $\{z_n\}$ converges.

One of the most important notions in the analysis (real or complex) is a theory that allows us to add up infinitely many terms. To make sense of such an idea we begin with a sequence $\{z_n\}$, and form a new sequence $\{S_n\}$, called the sequence of partial sums, as follows.

$$S_1 = z_1,$$

$$S_2 = z_1 + z_2,$$

$$S_3 = z_1 + z_2 + z_3,$$

$$\vdots$$

$$S_n = z_1 + z_2 + \ldots + z_n = \sum_{k=1}^{\infty} z_k,$$

$$\vdots$$

Definition 4.4 (Infinite Series). The formal expression $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \ldots + z_n + \ldots$ is called an infinite series, and $z_1, z_2, \ldots, z_n, \ldots$, are called the terms of the series.

If there is a complex number S for which $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} z_k$, we will say that the infinite series $\sum_{k=1}^{\infty} z_k$ converges to S, and that S is the sum of the infinite series. When this occurs, we write $S = \sum_{k=1}^{\infty} z_k$. The series $\sum_{k=1}^{\infty} z_k$ is said to be absolutely convergent provided that the (real) series of magnitudes $\sum_{k=1}^{\infty} |z_k|$ converges. If a series does not converge, we say that it diverges.

Remark 4.2. The first finitely many terms of a series do not affect its convergence or divergence and, in this respect, the beginning index of a series is irrelevant. Thus, we will without comment conclude that if a series $\sum_{k=N+1}^{\infty} z_k$ converges, then so does $\sum_{k=1}^{\infty} z_k$, where $z_1, z_2, \ldots, z_N$ is any finite collection of terms. A similar remark holds for determining the divergence of a series.

As you might expect, many of the results concerning real series carry over to complex series. We now give several of the more standard theorems for complex series, along with examples of how they are used.

Theorem 4.4. Let $z_n = x_n + i\, y_n$ and $S = U + i\, V$. Then

$$S = \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (x_n + i\, y_n)$$ (converges) if and only if

both $$U = \sum_{n=1}^{\infty} x_n \text{ and } V = \sum_{n=1}^{\infty} y_n$$ (converge).

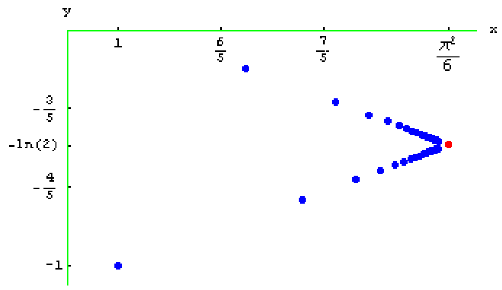
Theorem 4.5. If $\sum_{n=1}^{\infty} z_n$ is a convergent complex series, then $\lim_{n\to\infty} z_n = 0$.

Example 4.3. Show that the series

$$\sum_{n=1}^{\infty} \frac{1 + i\, n\, (-1)^n}{n^2} = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + i\, \frac{(-1)^n}{n}\right]$$ is convergent.

Solution. Recall that the real series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ are convergent. Hence, Theorem 4.4 implies that the given complex series is convergent.

Aside. Just for fun, we can graph some of the partial sums of this complex series.



The partial sums $$\{S_n\} = \left\{\sum_{k=1}^{n} \frac{1 + i\, k\, (-1)^k}{k^2}\right\}$$ converge to the value

$$S = \frac{\pi^2}{6} - i\, \ln(2).$$

The real and imaginary parts are

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{n\,(-1)^n}{n^2} = -\text{Log}[2]$$

The complex series is

$$\sum_{n=1}^{\infty}\frac{1+i\,n\,(-1)^n}{n^2} = \frac{\pi^2}{6} - i\,\text{Log}[2]$$

$$\sum_{n=1}^{\infty}\frac{1+i\,n\,(-1)^n}{n^2} = 1.64493 - 0.693147\,i$$

$$S_n = \left(\sum_{k=1}^{n}\frac{1}{k^2}\right) + i\left(\sum_{k=1}^{n}\frac{(-1)^k}{k}\right)$$

$$\{S_n\} = \left\{1-i,\ \frac{5}{4}-\frac{i}{2},\ \frac{49}{36}-\frac{5\,i}{6},\ \frac{205}{144}-\frac{7\,i}{12},\ \frac{5269}{3600}-\frac{47\,i}{60},\ \frac{5369}{3600}-\frac{37\,i}{60},\ \frac{266681}{176400}-\frac{319\,i}{420},\ \frac{1077749}{705600}-\frac{533\,i}{840},\ \frac{9778141}{6350400}-\frac{1879\,i}{2520},\ \frac{1968329}{1270080}-\frac{1627\,i}{2520},\ \ldots\right\}$$

$$\{S_n\} = \{1.-1.\,i,\ 1.25-0.5\,i,\ 1.36111-0.833333\,i,\ 1.42361-0.583333\,i,\ 1.46361-0.783333\,i,\ 1.49139-0.616667\,i,\ 1.5118-0.759524\,i,\ 1.52742-0.634524\,i,\ 1.53977-0.745635\,i,\ 1.54977-0.645635\,i,\ \ldots\}$$

$$S_n = \left(\sum_{k=1}^{n}\frac{1}{k^2}\right) + i\left(\sum_{k=1}^{n}\frac{(-1)^k}{k}\right) = \text{HarmonicNumber}[n,2] + i\left(-\frac{\text{Log}[4]}{2} + \frac{1}{2}(-1)^n\left(\text{PolyGamma}\left[0, 1+\frac{n}{2}\right] - \text{PolyGamma}\left[0, \frac{1+n}{2}\right]\right)\right)$$

The sum of the infinite series is

$$S = \sum_{k=1}^{\infty}\frac{1}{k^2} + i\sum_{k=1}^{\infty}\frac{(-1)^k}{k} = \frac{\pi^2}{6} - i\,\text{Log}[2]$$

$$S_n = S(n) = \text{HarmonicNumber}[n,2] + i\left(-\frac{\text{Log}[4]}{2} + \frac{1}{2}(-1)^n\left(\text{PolyGamma}\left[0, 1+\frac{n}{2}\right] - \text{PolyGamma}\left[0, \frac{1+n}{2}\right]\right)\right)$$

$$\lim_{n\to\infty} S_n = \frac{\pi^2}{6} - i\,\text{Log}[2]$$

$$\lim_{n\to\infty} S_n = 1.64493 - 0.693147\,i$$

$$S = \sum_{k=1}^{\infty}\frac{1+i\,k\,(-1)^k}{k^2} = \frac{\pi^2}{6} - i\,\text{Log}[2]$$

$$S = \sum_{k=1}^{\infty}\frac{1+i\,k\,(-1)^k}{k^2} = 1.64493 - 0.693147\,i$$

Therefore, we see that the infinite series converges, indeed

$\sum_{n=1}^{\infty}\frac{1+i\,n\,(-1)^n}{n^2} = \frac{\pi^2}{6} - i\,\text{Log}[2]$. **Example 4.4.** Show that the series

$\sum_{n=1}^{\infty}\frac{(-1)^n+i}{n} = \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n} + i\,\frac{1}{n}\right]$ is divergent.

Solution. We know that the real series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Hence, Theorem 4.4 implies that the given complex series is divergent.

Notes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\text{Log}[2]$$

The real part converges.

Sum::div : Sum does not converge.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

The imaginary part diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n + i}{n} = -\text{Log}[2] + i \sum_{n=1}^{\infty} \frac{1}{n}$$

The series diverges.

Remark 1. Since the answer returned for the imaginary part was $\sum_{n=1}^{\infty} \frac{1}{n}$, this means that a sum was not found. It is known that the partial sums of the harmonic series grow slowly without bound. For example, adding 10, 100, 1000 and 10000 terms yields:

Remark 2. The integral test could also be used.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} dx = \lim_{b \to \infty} \text{Log}[b] = \infty$$

The integral diverges, therefore

the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞.

$$\{S_n\} = \left\{-1 + i, -\frac{1}{2} + \frac{3 i}{2}, -\frac{5}{6} + \frac{11 i}{6}, -\frac{7}{12} + \frac{25 i}{12}, -\frac{47}{60} + \frac{137 i}{60}, -\frac{37}{60} + \frac{49 i}{20}, -\frac{319}{420} + \frac{363 i}{140}, -\frac{533}{840} + \frac{761 i}{280}, -\frac{1879}{2520} + \frac{7129 i}{2520}, -\frac{1627}{2520} + \frac{7381 i}{2520}, \ldots\right\}$$

$$\{S_n\} = \{-1. + 1. i, -0.5 + 1.5 i, -0.833333 + 1.83333 i, -0.583333 + 2.08333 i, -0.783333 + 2.28333 i, -0.616667 + 2.45 i, -0.759524 + 2.59286 i, -0.634524 + 2.71786 i, -0.745635 + 2.82897 i, -0.645635 + 2.92897 i, \ldots\}$$

$$S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k} + i \sum_{k=1}^{n} \frac{1}{k} = i\,\text{HarmonicNumber}[n] - \frac{\text{Log}[4]}{2} + \frac{1}{2} (-1)^n \left(\text{PolyGamma}\left[0, 1 + \frac{n}{2}\right] - \text{PolyGamma}\left[0, \frac{1+n}{2}\right]\right)$$

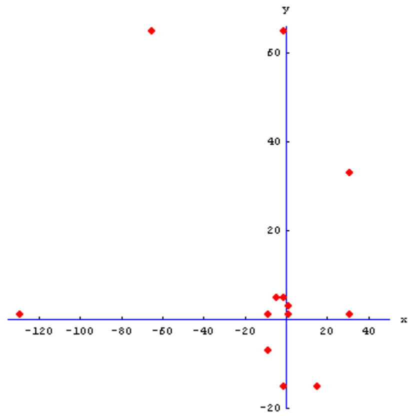
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} + i \sum_{k=1}^{\infty} \frac{1}{k} = \text{Indeterminate}$$

Example 4.5. Show that the series $\sum_{n=1}^{\infty} (1 + i)^n$ is divergent.

Solution. Here we set $z_n = (1+i)^n$ and observe that

$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |(1+i)^n| = \lim_{n\to\infty} \left(\sqrt{2}\right)^n = \infty$. Thus $\lim_{n\to\infty} z_n \neq 0$, and Theorem 4.5 implies that the series is not convergent; hence it is divergent.

Aside. Just for fun, we can graph some of the partial sums of this divergent complex series.



The sequence of partial sums $\{S_n\} = \left\{\sum_{k=1}^{n} (1+i)^k\right\}$ diverges.

$$\lim_{n\to\infty} |(1+i)^n| = \lim_{n\to\infty} \mathrm{Abs}[(1+i)^n]$$

$$\lim_{n\to\infty} |(1+i)^n| = \lim_{n\to\infty} 2^{n/2}$$

$$\lim_{n\to\infty} |(1+i)^n| = \infty$$

Hence $\lim_{n\to\infty} z_n \neq 0$, and the series is divergent.

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 State Dirichlet Problem.

Solution :

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Notes

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Q. 2 Define Extended Fourier in The Unit Disk.

Solution :

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Theorem 4.6. Let $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ be convergent series, and let c be a complex number. Then $\sum_{n=1}^{\infty} c\, z_n = c \sum_{n=1}^{\infty} z_n$

and $\sum_{n=1}^{\infty} (z_n + w_n) = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$.

Definition 4.5 (Cauchy Product of Series). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series, where a_n and b_n are complex numbers. The Cauchy product of the two series is defined to be the series $\sum_{n=1}^{\infty} c_n$, where $c_n = \sum_{k=1}^{n} a_k\, b_{n-k}$.

Theorem 4.7. If the Cauchy product converges, then

$\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$, where $c_n = \sum_{k=1}^{n} a_k\, b_{n-k}$.

The proof can be found in a number of texts, for example, Infinite Sequences and Series, by Konrad Knopp (translated by Frederick Bagemihl; New York: Dover, 1956).

Theorem 4.8 (Comparison Test). Let $\sum_{n=1}^{\infty} M_n$ be a convergent series of real nonnegative terms. If $\{z_n\}$ is a sequence of complex numbers and $|z_n| \le M_n$ holds for all n, then $\sum_{n=1}^{\infty} z_n$ converges.

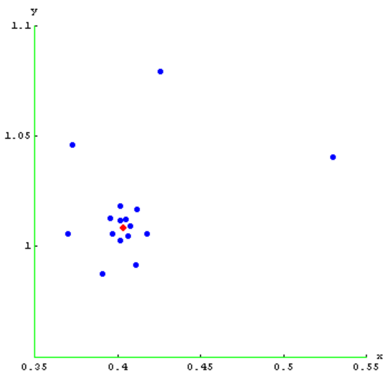
Corollary 4.1. If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges.

In other words, absolute convergence implies convergence for complex series as well as for real series.

Example 4.6. Show that the series $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ is convergent.

Solution. We calculate $|z_n| = \left| \frac{(3+4i)^n}{5^n n^2} \right| = \frac{1}{n^2} = M_n$. Using the comparison test and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we determine that $\sum_{n=1}^{\infty} \left| \frac{(3+4i)^n}{5^n n^2} \right|$ converges and hence, by Corollary 4.1, so does $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$.

Aside. Just for fun, we can graph some of the partial sums of this complex series.



The partial sums $\{S_n\} = \left\{ \sum_{k=1}^{n} \frac{(3+4i)^k}{5^k k^2} \right\}$ converge to the value

$S \approx 0.403311 + 1.00841\, i$

6 The Cauchy Criterion (General Principle of Convergence)

A sequence of real numbers, a_1, a_2,... , a_n,... will have a finite limit value or will be convergent if for no matter how small a positive number □ we take there exists a term a_n such that the distance between that term and every term further in the sequence is smaller than □, that is, by moving further in the sequence the difference between any two terms gets smaller and smaller.

As a_{n+r}, where $r = 1, 2, 3,...$ denotes any term that follows a_n, then

$$| a_{n+r} < a_n | < \text{ for all } n > n_0, \quad r = 1, 2, 3,...$$

shows the condition for the convergence of a sequence.

If a sequence $\{a_n\}$ of real numbers (or points on the real line) the distances between which tend to zero as their indices tend to infinity, then $\{a_n\}$ is a Cauchy sequence.

Therefore, if a sequence $\{a_n\}$ is convergent, then $\{a_n\}$ is a Cauchy sequence.

The Cauchy criterion or general principle of convergence, example

The following example shows us the nature of that condition.

Example: We know that the sequence 0.3, 0.33, 0.333,... converges to the number 1/3 as

1/3 = 0.33333.... Let write the rule for the n^{th} term,

$$0.3,\ 0.33,\ 0.333, \dots \quad \text{or} \quad \frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \dots$$

$$\text{or} \quad \frac{3}{10}\cdot 1,\ \frac{3}{10^2}\cdot 11,\ \frac{3}{10^3}\cdot 111, \dots,\ \frac{3}{10^n}\cdot\frac{10^n-1}{9}, \dots, \quad \text{so that} \quad a_n = \frac{1}{3}\cdot\left(1-\frac{1}{10^n}\right).$$

If we go along the sequence far enough, say to the 100^{th} term, i.e., the term with a hundred 3's in the fractional part, then the difference between that term and every next term is equal to the decimal fraction with the fractional part that consists of a hundred 0's followed by 3's on the lower decimal places, starting from the 101st decimal place. That is,

$$|a_{101} - a_{100}| = \left| \frac{1}{3} \cdot \left(1 - \frac{1}{10^{101}}\right) - \frac{1}{3} \cdot \left(1 - \frac{1}{10^{100}}\right) \right| = \left| -\frac{1}{3} \cdot \frac{1}{10^{101}} + \frac{1}{3} \cdot \frac{1}{10^{100}} \right| = \frac{1}{3} \cdot \frac{1}{10^{100}} \left(1 - \frac{1}{10}\right) = \frac{3}{10^{101}}.$$

Therefore, the absolute value of the difference falls under	$\frac{1}{10^{100}}.$

Then, if we go further along the sequence and for example calculate the distance between the 100000th term

and the following terms, the distance will be smaller than	$\frac{1}{10^{100000}}.$

Hence, since we can make the left side of the inequality as small as we wish by choosing n large enough, then all terms that follow a_n (denoted a_{n+r}, $r = 1, 2, 3, ...$), infinitely many of them, lie in the interval of the length 2 symmetrically around the point a_n. Outside of that interval, there is only a finite number of terms. That is,

$$< a_{n+r} \quad a_n < + \text{ for all } n > n_0 \text{ , } r = 1, 2, 3, ...$$

or $a_n < a_{n+r} < a_n + 1$.

So, the terms of the sequence, starting from the $(n + 1)^{th}$ term, form the infinite and bounded sequence of numbers and so, according to the above theorem, they must have at least one cluster point that lies in that interval. But they cannot have more than one cluster point since all points that follow the n^{th} term lie inside the interval 2 lengths of which is arbitrarily small if n is already large enough so that any other cluster point will have to be outside of that interval.

Thus, the above theorem simply says that if a sequence converges, then the terms of the sequence are getting closer and closer to each other as shown in the example.

Some important limits

(1) Let examine the convergence of the sequence given by $a_n = |\, a \,|^n$

a) if $|\, a \,| > 1$ then we can write $|\, a \,| = 1 + h$, where h is a positive number.

So, by the binomial theorem

$$|\, a \,|^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{1 \cdot 2} h^2 + \cdots + h^n.$$

If we drop all terms beginning from the third that are all positive since the binomial coefficients are natural numbers, if $n > 2$ and $h > 0$, the right side become smaller, so obtained is the Bernoulli's inequality

$$(1 + h)^n > 1 + nh, \;\; n > 2.$$

When n then $1 + nh$ tends to the positive infinity too, since we can make $1 + nh$ greater than any

given positive number N, if only we take	$n > \frac{N-1}{h}$,	therefore will even more tend to infinity $|\, a \,|^n$ which is

greater. Thus,

$$\lim_{n\to\infty} |a|^n = \infty, \quad |a| > 1,$$ or $$\lim_{n\to\infty} a^n = \infty \quad \text{if} \quad a > 1.$$

b) if $0 < |a| < 1$ then we can write $$|a| = \frac{1}{b},\ b > 1 \ \text{ and } \ |a|^n = \frac{1}{b^n}.$$

Since $b > 1$ then $\lim_{n\to\infty} b^n = \infty$, such that $|a|^n <$ whenever $\frac{1}{b^n} < \varepsilon$ or $b^n > \frac{1}{\varepsilon}$, however small □ is,

this inequality can be satisfied by choosing *n* large enough. Therefore,

$$\lim_{n\to\infty} |a|^n = 0, \quad |a| < 1,$$ or $$\lim_{n\to\infty} a^n = 0, \quad \text{if} \quad 0 < a < 1.$$

(2) Let examine convergence of the sequence give by $$a_n = \frac{|a|^n}{n!}.$$

The se uence $$|a|,\ \frac{|a|}{2!},\ \frac{|a|}{3!},\ \cdots,\ \frac{|a|}{n!},\ \cdots,\ a \in \mathbb{R},$$ the *n*-th term of which we can write as

$$\frac{|a|^n}{n!} = \frac{|a|}{1}\cdot\frac{|a|}{2}\cdot\frac{|a|}{3}\cdot\ldots\cdot\frac{|a|}{m}\cdot\frac{|a|}{m+1}\cdot\ldots\cdot\frac{|a|}{n}.$$

For every $|a| > 1$ there exists a natural number m such that $m < |a| < m + 1$ and $n > m$ then

$$\frac{|a|^n}{n!} = \frac{|a|^n}{m!} \cdot \frac{|a|^{n-m}}{(m+1)\cdot(m+2)\cdot\ldots\cdot n} \leq \frac{|a|^m}{m!} \cdot \frac{|a|^{n-m}}{(m+1)^{n-m}},$$

since $\frac{|a|}{m+1} < 1$ it follows that $a_n < 0$ or we write

$$\boxed{\lim_{n\to\infty} \frac{|a|^n}{n!} = 0, \quad \text{for all} \quad n \in \mathbb{N} \text{ and } a \in \mathbb{R}.}$$

(3) Let examine convergence of the sequence given by

$$a_n = \sqrt[n]{a} = a^{\frac{1}{n}}.$$

a) If $a > 1$ then the sequence $a, \sqrt{a}, \sqrt[3]{a}, \ldots, \sqrt[n]{a}, \ldots$ is decreasing, that is

$$a > \sqrt{a} > \sqrt[3]{a} > \cdots > \sqrt[n]{a} > \sqrt[n+1]{a} > \cdots > 1.$$

Let $\sqrt[n]{a} = 1 + h, \quad h > 0 \quad \text{or} \quad a = (1+h)^n$ then by the Bernoulli's inequality $a > 1 + nh$ so that

$$0 < h < \frac{a-1}{n}.$$

Since the numerator $a < 1$ is fixed number then, if $n < 0$ then $h < 0$ too, therefore $\sqrt[n]{a} = 1 + h$ tends to 1. So we can write

$$\lim_{n\to\infty} \sqrt[n]{a} = 1, \quad \text{for all } a > 1.$$

b) If $0 < a < 1$ then the $a, \sqrt{a}, \sqrt[3]{a}, \ldots, \sqrt[n]{a}, \ldots$ is increasing, sequence that is

$$a < \sqrt{a} < \sqrt[3]{a} < \cdots < \sqrt[n]{a} < \sqrt[n+1]{a} \cdots < 1.$$

For example,

$$0.04,\ \sqrt{0.04},\ \sqrt[3]{0.04},\ \cdots,\ \sqrt[10]{0.04},\ \cdots$$

$$\text{or}\quad 0.04,\quad 0.2,\quad 0.341995\ldots,\ 0.724779\ldots,\ \cdots$$

If we write $a = \frac{1}{b},\ b > 1$ then $\sqrt[n]{a} = \frac{1}{\sqrt[n]{b}}$ and $\sqrt[n]{b} \to 1$ as $n \to \infty$, that is $|1 - \sqrt[n]{b}| \to 0.$

Since $\sqrt[n]{b} > 1$ or $\frac{1}{\sqrt[n]{b}} < 1$ then $|\sqrt[n]{a} - 1| = \left|\frac{1}{\sqrt[n]{b}} - 1\right| = \frac{1}{\sqrt[n]{b}}|1 - \sqrt[n]{b}| < |1 - \sqrt[n]{b}|,$

so it follows that $|\sqrt[n]{a} - 1| \to 0$ as $n \to \infty.$ Therefore, $\lim_{n\to\infty} \sqrt[n]{a} = 1,\ 0 < a < 1.$

c) If $a = 1$ then $\sqrt[n]{a} = 1$ so that $\lim_{n\to\infty} \sqrt[n]{a} = 1,\ a = 1.$

Since in all three cases above, a), b) and c) we've got the same result, then we can write

$$\boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1,\ a > 0.}$$

5.6 SPECIFIC GEOMETRIC SERIES

- Grandi's series: 1 − 1 + 1 − 1 + ⋯
- 1 + 2 + 4 + 8 + ⋯
- 1 − 2 + 4 − 8 + ⋯
- 1/2 + 1/4 + 1/8 + 1/16 + ⋯

- 1/2 − 1/4 + 1/8 − 1/16 + ⋯
- 1/4 + 1/16 + 1/64 + 1/256 + ⋯

5.7 SUMMARY

We study in this unit Geometric series and its examples. We study convergence and divergence of a series. We study sequence and series. We study Cauchy Sequence Convergence. We study General Principal of Convergence.

5.8 KEYWORD

Convergence : The tendency of unrelated animals and plants to evolve superficially similar characteristics under similar environmental conditions

Geometric : Characterized by or decorated with regular lines and shapes

Paradox : A statement or proposition which, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems logically unacceptable or self-contradictory.

5.9 QUESTIONS FOR REVIEW

Q. 1 If $|z| < 1$, the series $\sum_{n=0}^{\infty} z^n$ converges to $f(z) = \frac{1}{1-z}$. That is, if $|z| < 1$ then $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots + z^k + \ldots = \frac{1}{1-z}$

Q. 2 Show that $\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n} = 1 - i$.

Q. 3 If $\sum_{n=0}^{\infty} \zeta_n$ is a complex series with the property that $\lim_{n\to\infty} \frac{|\zeta_{n+1}|}{|\zeta_n|} = L$, then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

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5.11 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 4

2 Check in Section 1

Check In Progress-II

Answer Q. 1 Check in section 4.3

2 Check in Section 4.2

UNIT 6 : PRINCIPAL OF CONVERGENCE

STRUCTURE

6.0 OBJECTIVE

- In this unit, we shall study about necessary conditions for convergence of both sequence and series.
- We shall study also Cauchy General Principal of Convergence.

We study Weierstrass Product Inequality of series.

6.1 INTRODUCTION :

There are some **necessary conditions for convergence** of both sequences and series. Sequences: If a sequence converges, then it is Cauchy.... If a series converges, then the sequence as That means that an infinite sum only converges if the terms of the sum are getting closer and closer to 0

6.1.1 Necessary Conditions for Convergence

As with sequences, the convergence of an infinite series	$\sum_{n=1}^{\infty} a_n$	only depends on the behavior of the

a general term of the series a_n as n increases to infinity, and not on any finite number of its initial terms.

Note that, since	$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n$, where $m, n \in \mathbb{N}$,	

the series	$\sum_{n=1}^{\infty} a_n$	converges if and only if	$\sum_{n=m}^{\infty} a_n$	converges.
Therefore, to show that a series	$\sum_{n=1}^{\infty} a_n$	converges we can ignore any finite number of terms	$\sum_{n=1}^{m-1} a_n$	at the

beginning, and just need to prove the convergence of the tail or remainder	$\sum_{n=m}^{\infty} a_n$	of the series.

The difference between the sum s of a convergent series $a_1 + a_2 + a_3 + ... + a_n + ...$ and the n^{th} partial sum s_n is called the remainder (tail) r_n of the series, i.e.,

$$r_n = s\ ,\ s_n = a_{n+1} + a_{n+2} + a_{n+3} + ... \quad \text{or } s = s_n + r_n.$$

Thus, if a series	$\sum_{n=1}^{\infty} a_n$	converges then the remainder $r_n = a_{n+1} + a_{n+2} + a_{n+3} + ...$ converges too,

that is, since	$\lim_{n\to\infty} s_n = s$	and s $= s_n + r_n$ then,	$\lim_{n\to\infty} r_n = 0.$	

Necessary and sufficient condition for the convergence of a series - Cauchy's convergence test

Necessary and sufficient condition that the sequence of partial sums $\{s_n\}$ of a given series converges, and

hence the series	$\sum_{n=1}^{\infty} a_n$	converges is, that for given however small positive number n , it is possible find

an index n_0 such that $| s_{n+r} , s_n | < \infty$ whenever $n \geq n_0$ and $r = 1, 2, 3,\ldots$,

or expressed by terms of the series, if

$| a_{n+1} + a_{n+2} + a_{n+3} + \ldots + a_{n+r} | < \infty$ whenever $n \geq n_0$ and $r = 1, 2, 3,\ldots.$

Therefore, a series converges if the absolute value of the sum of any finite number of sequential terms can become arbitrary small by starting the addition from a term that is far enough.

Necessary condition for the convergence of a series

Hence, it is a necessary condition for the convergence of a series that its terms tend to zero as n

increases to infinity, that is	$\lim_{n\to\infty} a_n = 0.$	So, if this condition is not satisfied with the series diverges.

That this condition is only necessary but not sufficient condition for the convergence shows the harmonic series for which

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{but} \quad s = \lim_{n\to\infty} s_n = \infty,$$

as was shown in the previous section.

The necessary condition for the convergence of a series is usually used to show that a series does not converge.

The nth term test for divergence

$$\text{If } \lim_{n\to\infty} a_n \text{ does not exist or if } \lim_{n\to\infty} a_n \neq 0, \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Note that, since	$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n$, where $m, n \in \mathrm{N}$,	

the series	$\sum_{n=1}^{\infty} a_n$	converges if and only if	$\sum_{n=m}^{\infty} a_n$	converges.

Therefore, to show that a series	$\sum_{n=1}^{\infty} a_n$	converges we can ignore any finite number of terms	$\sum_{n=1}^{m-1} a_n$	at the

beginning, and just need to prove the convergence of the tail or remainder	$\sum_{n=m}^{\infty} a_n$	of the series.

The difference between the sum s of a convergent series a1 + a2 + a3 +... + an +... and the nth partial sum sn is called the remainder (tail) rn of the series, i.e.,

rn = s - sn = an + 1 + an + 2 + an + 3 +... or s = sn + rn.

Thus, if a series	$\sum_{n=1}^{\infty} a_n$	converges then the remainder $r_n = a_{n+1} + a_{n+2} + a_{n+3} +$... converges too,

that is, since	$\lim_{n\to\infty} s_n = s$	and $s = s_n + r_n$ then,	$\lim_{n\to\infty} r_n = 0.$	

Necessary and sufficient condition for the convergence of a series - Cauchy's convergence test

Necessary and sufficient condition that the sequence of partial sums {sn} of a given series converges, and

hence the series	$\sum_{n=1}^{\infty} a_n$	converges is, that for given however small positive number □ it is possible to find

Note that this is only a test for divergence. That is, if we can prove that the sequence $\{a_n\}$ does not converge to 0, then the infinite series does not converge.

Properties of series

If given are two convergent series,

$$\sum_{n=1}^{\infty} a_n = s_a \text{ and } \sum_{n=1}^{\infty} b_n = s_b,$$

then the convergent series is obtained by adding or subtracting their same index terms, and its sum equals the sum or the difference of their individual sums, i.e.,

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = s_a \pm s_b.$$

If $\sum_{n=1}^{\infty} a_n = s$, then $\sum_{n=1}^{\infty}(c \cdot a_n) = c \cdot s$ for any constant c.

The product of two series or the Cauchy product

If given are two convergent series of positive terms,	$\sum_{n=1}^{\infty} a_n = s_a$ and $\sum_{n=1}^{\infty} b_n = s_b$,	then the product

$$s_a \cdot s_b = \sum_{n=1}^{\infty}(a_1 b_n + a_2 b_{n-1} + \cdots + a_{n-1} b_2 + a_n b_1)$$

denotes the convergent series sum of which is equal to the product of the sums of the given series.

6.2 THE CAUCHY CRITERION (GENERAL PRINCIPLE OF CONVERGENCE)

Sufficient condition for convergence of a sequence - The Cauchy criterion (general principle of convergence)

A sequence of real numbers, a_1, a_2,... , a_n,... will have a finite limit value or will be convergent if for no matter how small a positive number we take there exists a term a_n such that the distance between that term and every term further in the sequence is smaller than, that is, by moving further in the sequence the difference between any two terms gets smaller and smaller.

As a_{n+r}, where $r = 1, 2, 3,...$ denotes any term that follows a_n, then

$$| a_{n+r} , a_n | < \infty \text{ for all } n \geq n_0 , r = 1, 2, 3,...$$

shows the condition for the convergence of a sequence.

If a sequence $\{a_n\}$ of real numbers (or points on the real line) the distances between which tend to zero as their indices tend to infinity, then $\{a_n\}$ is a Cauchy sequence.

Notes

Therefore, if a sequence $\{a_n\}$ is convergent, then $\{a_n\}$ is a Cauchy sequence.

The Cauchy criterion or general principle of convergence, example

The following example shows us the nature of that condition.

Example: We know that the sequence 0.3, 0.33, 0.333,... converges to the number 1/3 as

1/3 = 0.33333.... Let write the rule for the n^{th} term,

$$0.3,\ 0.33,\ 0.333,\dots \quad \text{or} \quad \frac{3}{10},\ \frac{33}{100},\ \frac{333}{1000},\dots$$

$$\text{or} \quad \frac{3}{10}\cdot 1,\ \frac{3}{10^2}\cdot 11,\ \frac{3}{10^3}\cdot 111,\dots,\ \frac{3}{10^n}\cdot\frac{10^n-1}{9},\dots,\ \text{ so that } \quad a_n=\frac{1}{3}\cdot\left(1-\right.$$

If we go along the sequence far enough, say to the 100^{th} term, i.e., the term with a hundred 3's in the fractional part, then the difference between that term and every next term is equal to the decimal fraction with the fractional part that consists of a hundred 0's followed by 3's on the lower decimal places, starting from the 101^{st} decimal place. That is,

$$|a_{101}-a_{100}|=\left|\frac{1}{3}\cdot\left(1-\frac{1}{10^{101}}\right)-\frac{1}{3}\cdot\left(1-\frac{1}{10^{100}}\right)\right|=\left|-\frac{1}{3}\cdot\frac{1}{10^{101}}+\frac{1}{3}\cdot\frac{1}{10^{100}}\right|=\frac{1}{3}$$

Therefore, the absolute value of the difference falls under	$\frac{1}{10^{100}}$.

Then, if we go further along the sequence and for example calculate the distance between the 100000^{th} term

and the following terms, the distance will be smaller than	$\frac{1}{10^{100000}}$.

Hence, since we can make the left side of the inequality $| a_{n+r} , a_n | <$ as small as we wish by choosing n large enough, then all terms that follow a_n (denoted a_{n+r}, $r = 1, 2, 3,\dots$), infinitely many of them, lie in the interval of the length 2 symmetrically around the point a_n. Outside of that interval there is only a finite number of terms. That is,

$$< a_{n+r} , a_n < + \text{ for all } n \geq n_0 ,\ r = 1, 2, 3,\dots$$

$$\text{or} \quad a_n < a_{n+r} < a_n + .$$

So, the terms of the sequence, starting from the $(n + 1)^{\text{th}}$ term, form the infinite and bounded sequence of numbers and so, according to the above theorem, they must have at least one cluster point that lies in that interval. But they cannot have more than one cluster point since all points that follow the n^{th} term lie inside the interval 2 length of which is arbitrarily small, if n is already large enough so that any other cluster point will have to be outside of that interval.

Thus, the above theorem simply says that if a sequence converges, then the terms of the sequence are getting closer and closer to each other as shown in the example.

Some important limits

(1) Let examine convergence of the sequence given by $a_n = |\, a\,|^n$

a) if $|\, a\,| > 1$ then we can write $|\, a\,| = 1 + h$, where h is a positive number.

So, by the binomial theorem

$$|\, a\,|^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{1\cdot 2}h^2 + \cdots + h^n.$$

If we drop all terms beginning from the third that are all positive since the binomial coefficients are natural numbers, if $n \geq 2$ and $h > 0$, the right side become smaller, so obtained is the Bernoulli's inequality

$$(1 + h)^n > 1 + nh, \;\; n \geq 2.$$

When n then $1 + nh$ tends to the positive infinity too, since we can make $1 + nh$ greater than any

given positive number N, if only we take	$n > \dfrac{N-1}{h}$,	therefore will even more tend to infinity $\|\, a\,\|^n$ which is

greater. Thus,

$\lim\limits_{n\to\infty} \|a\|^n = \infty, \quad \|a\| > 1,$	or	$\lim\limits_{n\to\infty} a^n = \infty \quad \text{if} \quad a > 1.$	

b) if $0 < \|\, a\,\| < 1$ then we can write	$\|a\| = \dfrac{1}{b},\; b > 1 \;\text{ and }\; \|a\|^n = \dfrac{1}{b^n}.$	

Since $b >$ 1 then	$\lim\limits_{n\to\infty} b^n = \infty,$	such that $\|\, a\,\|^n <$ □ whenever	$\dfrac{1}{b^n} < \varepsilon \text{ or } b^n > \dfrac{1}{\varepsilon},$	however small □ is,

Notes

this inequality can be satisfied by choosing n large enough. Therefore,

$$\lim_{n\to\infty} |a|^n = 0, \quad |a| < 1, \quad \text{or} \quad \boxed{\lim_{n\to\infty} a^n = 0, \quad \text{if} \quad 0 < a < 1.}$$

(2) Let examine convergence of the sequence given by $a_n = \dfrac{|a|^n}{n!}$.

The sequence $|a|, \dfrac{|a|}{2!}, \dfrac{|a|}{3!}, \dots, \dfrac{|a|}{n!}, \dots,\ a \in \mathbb{R}$, the n-th term of whic we can write as

$$\frac{|a|^n}{n!} = \frac{|a|}{1} \cdot \frac{|a|}{2} \cdot \frac{|a|}{3} \cdot \ldots \cdot \frac{|a|}{m} \cdot \frac{|a|}{m+1} \cdot \ldots \cdot \frac{|a|}{n}.$$

For every $|a| \geq 1$ there exists a natural number m such that $m \leq |a| < m + 1$ and $n > m$ then

$$\frac{|a|^n}{n!} = \frac{|a|^n}{m!} \cdot \frac{|a|^{n-m}}{(m+1)\cdot(m+2)\cdot \ldots \cdot n} \leq \frac{|a|^m}{m!} \cdot \frac{|a|^{n-m}}{(m+1)^{n-m}},$$

since $\dfrac{|a|}{m+1} < 1$ it follows that a_n □ 0 or we write

$$\boxed{\lim_{n\to\infty} \frac{|a|^n}{n!} = 0, \quad \text{for all} \quad n \in \mathbb{N} \ \text{and} \ a \in \mathbb{R}.}$$

(3) Let examine convergence of the sequence given by $a_n = \sqrt[n]{a} = a^{\frac{1}{n}}$.

a) If $a > 1$ then the sequence $a, \sqrt{a}, \sqrt[3]{a}, \dots, \sqrt[n]{a}, \dots$ is decreasing, that is

$$a > \sqrt{a} > \sqrt[3]{a} > \cdots > \sqrt[n]{a} > \sqrt[n+1]{a} > \cdots > 1.$$

Let $\sqrt[n]{a} = 1 + h, \quad h > 0 \quad \text{or} \quad a = (1+h)^n$ then by the Bernoulli's inequality $a > 1 + nh$ so that

$0 < h < \dfrac{a-1}{n}.$ Since the numerator, a □ 1 is fixed number then, if n □ 0 then h □ 0 too, therefore

$\sqrt[n]{a} = 1 + h$ tends to 1. So we can write $\lim_{n\to\infty} \sqrt[n]{a} = 1, \quad \text{for all} \quad a > 1.$

b) If $0 < a < 1$ then the sequence $a, \sqrt{a}, \sqrt[3]{a}, \dots, \sqrt[n]{a}, \dots$ is increasing, that is

$$a < \sqrt{a} < \sqrt[3]{a} < \cdots < \sqrt[n]{a} < \sqrt[n+1]{a} \cdots < 1.$$

For example, $0.04, \sqrt{0.04}, \sqrt[3]{0.04}, \cdots, \sqrt[10]{0.04}, \cdots$

or $0.04, \quad 0.2, \quad 0.341995\dots, 0.724779\dots, \cdots$

If we write $a = \frac{1}{b}, \; b > 1$ then $\sqrt[n]{a} = \frac{1}{\sqrt[n]{b}}$ and $\sqrt[n]{b} \to 1$ as $n \to \infty$, that is $|1-$

Since $\sqrt[n]{b} > 1$ or $\frac{1}{\sqrt[n]{b}} < 1$ then $|\sqrt[n]{a} - 1| = \left| \frac{1}{\sqrt[n]{b}} - 1 \right| = \frac{1}{\sqrt[n]{b}} |1 - \sqrt[n]{b}| < |1$

so it follows that $|\sqrt[n]{a} - 1| \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \sqrt[n]{a} = 1, \quad 0 < a < 1.$

c) If $a = 1$ then $\sqrt[n]{a} = 1$ so that $\lim_{n\to\infty} \sqrt[n]{a} = 1, \quad a = 1.$

Since in all three cases above, a), b) and c) we've got the same result, then we can write

$$\boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1, \quad a > 0.}$$

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 Define the necessary conditions for convergence.

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

Q. 2 State General Principal of Convergence.

Solution :

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………

6.3 WEIERSTRASS PRODUCT INEQUALITY

In mathematics, the **Weierstrass** product **inequality** states that, For given real numbers $0 \leq a_1, a_2, a_3, a_4, \ldots, a_n \leq 1$. Where, The **inequality** is named after the German mathematician Karl **Weierstrass**. It can be proved by mathematical induction

If $0 \leq a, b, c, d \leq 1$, then

$$(1-a)(1-b)(1-c)(1-d) + a + b + c + d \geq 1.$$

This is a special case of the general inequality

$$\prod_{i=1}^{n}(1-a_i) + \sum_{i=1}^{n} a_i \geq 1$$

for $0 \leq a_1, a_2, \ldots, a_n \leq 1$. This can be proved by induction by supposing the inequality is true for $n = k$ and then adding a new element z. The sum then increases by z, while the product p increases by $(1-z)\,p - p$. The total increase is then $z + (1-z)\,p - p = z\,(1-p)$, which is greater than 0 since both z and $1-p$ are between 0 and 1. Since the inequality is true for $n = 1$ ($1 - a_1 + a_1 = 1 \geq 1$), it is therefore true for all n.

Geometric Series and Convergence Theorems

We begin this section by presenting a series of the form $\sum_{n=0}^{\infty} z^n$, which is called a geometric series and is one of the most important series in mathematics.

Theorem (Geometric Series). If $|z| < 1$, the series $\sum_{n=0}^{\infty} z^n$ converges to $f(z) = \frac{1}{1-z}$. That is, if $|z| < 1$ then

$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... + z^k + ... = \frac{1}{1-z}$. If $|z| \geq 1$, the series diverges.

Corollary 2. If $|z| > 1$, the series $\sum_{n=1}^{\infty} z^{-n}$ converges to $f(z) = \frac{1}{z-1}$.

That is, if $|z| > 1$ then $\sum_{n=1}^{\infty} z^{-n} = z^{-1} + z^{-2} + ... + z^{-4} + ... = \frac{1}{z-1}$,

or equivalently, $-\sum_{n=1}^{\infty} z^{-n} = -z^{-1} - z^{-2} - ... - z^{-4} - ... = \frac{1}{1-z}$. If $|z| \leq 1$, the series diverges.

Corollary. If $z \neq 1$, then for all n,

$\frac{1}{1-z} = 1 + z + z^2 + ... + z^{n-1} + \frac{z^n}{1-z}$.

Example. Show that $\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n} = 1 - i$.

Solution. If we set $z = \frac{1-i}{2}$, then $|z| = \left|\frac{1-i}{2}\right| = \frac{\sqrt{2}}{2} < 1$. By Theorem 4.12, the sum is $1 \Big/ \left(1 - \frac{1-i}{2}\right) = \frac{2}{2-1+i} = \frac{2}{1+i} = 1 - i$.

Solution:

We can use the definition of convergence of a series and find the limit of the partial sums.

$$z_k = \left(\frac{1}{2} - \frac{i}{2}\right)^k$$

$$S_n = \sum_{k=0}^{n} \frac{(1-i)^k}{2^k} = i\left((-1-i) + \left(\frac{1}{2} - \frac{i}{2}\right)^n\right)$$

$$\lim_{n\to\infty} S_n = 1 - i$$

$$\sum_{k=0}^{\infty} \frac{(1-i)^k}{2^k} = 1 - i$$

Or we can see that this is an infinite geometric series with ratio
$Z = \frac{1}{2} - \frac{i}{2}$.

$$z_k = \left(\frac{1}{2} - \frac{i}{2}\right)^k$$
$$z_{k+1} = \left(\frac{1}{2} - \frac{i}{2}\right)^{1+k}$$

$$Z = \frac{z_{k+1}}{z_k} = \frac{1}{2} - \frac{i}{2}$$

The sum of the infinite geometric series is now found.

$$Z = \frac{1}{2} - \frac{i}{2}$$
$$|Z| = \frac{1}{\sqrt{2}}$$
$$\text{Is } |Z| < 1 \text{ ? True}$$

$$\frac{1}{1-Z} = \frac{1}{1-\left(\frac{1}{2}-\frac{i}{2}\right)}$$
$$\frac{1}{1-Z} = 1 - i$$
$$\sum_{k=0}^{\infty} \frac{(1-i)^k}{2^k} = 1 - i$$

The series of absolute values converges, therefore the series converges.

We see that the sum of the infinite geometric series
$\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n}$ is $1 - i$.

Example. Evaluate $\sum_{n=3}^{\infty} \frac{i^n}{2^n}$.

Solution. We can put this expression in the form of a geometric series:

$$\sum_{n=3}^{\infty} \frac{i^n}{2^n} = \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n = \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^3 \left(\frac{i}{2}\right)^{n-3}$$

$$= \left(\frac{i}{2}\right)^3 \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^{n-3} = \left(\frac{i}{2}\right)^3 \frac{1}{1-\frac{i}{2}}$$

$$= -\frac{i}{8} \frac{1}{1-\frac{i}{2}} = \frac{-i}{8-4i}$$

$$= \frac{-i}{8-4i} \frac{8+4i}{8+4i} = \frac{4-8i}{80}$$

$$= \frac{1}{20} - \frac{i}{10}$$

We can use the definition of convergence of a series and find the limit of the partial sums.

$$z_k = \left(\frac{i}{2}\right)^k$$

$$S_n = \sum_{k=3}^{n} \frac{i^k}{2^k} = \frac{1}{20} - \frac{i}{10}$$

$$\lim_{n\to\infty} S_n = \frac{1}{20} - \frac{i}{10}$$

$$\sum_{k=3}^{\infty} \frac{i^k}{2^k} = \frac{1}{20} - \frac{i}{10}$$

Or we can see that this is an infinite geometric series with ratio $Z = \frac{i}{2}$.

$$a = \left(\frac{i}{2}\right)^3 = -\frac{i}{8}$$

$$Z = \frac{i}{2}$$

$$|Z| = \frac{1}{2}$$

Is $|Z| < 1$? True

$$\frac{a}{1-Z} = \frac{\left(\frac{i}{2}\right)^3}{1-\frac{i}{2}}$$

$$\frac{a}{1-Z} = \frac{1}{20} - \frac{i}{10}$$

$$\sum_{k=3}^{\infty} \frac{i^k}{2^k} = \frac{1}{20} - \frac{i}{10}$$

Notes

We see that the sum of the infinite geometric series

$\sum_{n=3}^{\infty} \frac{i^n}{2^n}$ is $\frac{1}{20} - \frac{i}{10}$.

Remark 4.3. The equality given in Example 4.14 illustrates an important point when evaluating a geometric series whose beginning index is other than zero. The value of $\sum_{n=r}^{\infty} z^n$ equals $\frac{z^r}{1 - z}$. If we think of z as the "ratio" by which a given term of the series is multiplied to generate successive terms, we see that the sum of a geometric series equals $\frac{\text{first term}}{1 - \text{ratio}}$, provided $|\text{ratio}| < 1$.

The geometric series is used in the proof of Theorem 4.12, which is known as the ratio test. It is one of the most commonly used tests for determining the convergence or divergence of series. The proof is similar to the one used for real series, and we leave it for you to do.

6.4 D'ALEMBERT'S RATIO TEST

Theorem (d'Alembert's Ratio Test). If $\sum_{n=0}^{\infty} \zeta_n$ is a complex series with the property that $\lim_{n\to\infty} \frac{|\zeta_{n+1}|}{|\zeta_n|} = L$, then the series converges absolutely if $L < 1$ and diverges if $L > 1$

Example. Show that $\sum_{n=0}^{\infty} \frac{(1 - i)^n}{n!}$ converges.

Solution. Using the ratio test, we find that

$$\lim_{n\to\infty}\left|\frac{(1-i)^{n+1}}{(n+1)!}\Big/\frac{(1-i)^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{n!\,(1-i)}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!\,(1-i)}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\frac{n!\,|1-i|}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{n!\,|1-i|}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{|1-i|}{n+1}$$

$$= \lim_{n\to\infty}\frac{\sqrt{2}}{n+1}$$

$$= 0 = L$$

Because $L < 1$, the series converges.

Solution.

Enter the formula for the terms in the series.

$$z_n = \frac{(1-i)^n}{n!}$$

$$z_{n+1} = \frac{(1-i)^{1+n}}{(1+n)!}$$

$$z_n = \frac{(1-i)^n}{n!}$$

$$\frac{z_{n+1}}{z_n} = \frac{(1-i)\,n!}{(1+n)!}$$

$$\frac{z_{n+1}}{z_n} = \frac{1-i}{1+n}$$

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{\sqrt{2}}{1+n}$$

$$L = \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0$$

Since L < 1, the series converges.

We see that the infinite series $\sum_{n=0}^{\infty} \frac{(1-i)^n}{n!}$ converges and that its sum is e^{1-i}.

Example Show that $\sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n}$ converges for all z in the disk $|z-i| < 2$.

Solution. Using the ratio test, we find that

$\lim_{n\to\infty} \left| \frac{(z-i)^{n+1}}{2^{n+1}} \Big/ \frac{(z-i)^n}{2^n} \right| == \lim_{n\to\infty} \left| \frac{z-i}{2} \right| == \frac{|z-i|}{2} == L$. If $|z-i| < 2$, then $\frac{|z-i|}{2} = L < 1$, and the series converges. If $|z-i| > 2$, then $L > 1$, and the series diverges.

Solution.

Enter the formula for the terms in the series.

$$z_n = 2^{-n} (-i + z)^n$$

$$z_{n+1} = 2^{-1-n} (-i + z)^{1+n}$$
$$z_n = 2^{-n} (-i + z)^n$$

$$\frac{z_{n+1}}{z_n} = \frac{1}{2} (-i + z)$$

$$L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \frac{1}{2} \text{Abs}[-i + z]$$

When L < 1, the series will converge. Solve $\left| \frac{z-i}{2} \right| < 1$ and obtain the disk $|z-i| < 2$.

$$f[z] = \sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n} = -\frac{2}{(-2-i)+z}$$

We can investigate the convergence by plotting several partial sums of this series. Since convergence will be more rapid in a smaller disk $|z-i| < r$, the following plot will be a smaller disk with $r = 1$.

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 State d'Alembert Ratio Test.

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

Q. 2 State **Weierstrass** product **inequality**.

Solution :

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…………………………………………………………………………………

…………………………………………………………………………………

6.5 LIMIT SUPREMUM

Definition (Limit Supremum). Let $\{t_n\}$ be a sequence of positive real numbers. The limit supremum of the sequence (denoted by $\limsup_{n\to\infty} t_n$) is the smallest real number L with the property that for any $\epsilon > 0$ there are at most finitely many terms in the sequence that are larger than $L + \epsilon$. If there is no such number L, then we set $\limsup_{n\to\infty} t_n = \infty$

Limit Supremum: Given a sequence of real numbers a_n, the supremum limit (also called the limit superior or upper limit), written $\limsup$ and pronounced 'lim-soup,' is the limit of

$$A_n = \sup_{k\ge n} a_k$$

as $n \to \infty$, where $\sup_{x\in S} x$ denotes the supremum. Note that, by definition, A_n is nonincreasing and so either has a limit or tends to $-\infty$. For example, suppose $a_n = (-1)^n / n$, then for n odd, $A_n = 1/(n+1)$, and for n even, $A_n = 1/n$. Another example is $a_n = \sin n$, in which case A_n is a constant sequence $A_n = 1$.

When $\limsup a_n = \liminf a_n$, the sequence converges to the real number

Notes

$\lim a_n = \lim\sup a_n = \lim\inf a_n$.

Otherwise, the sequence does not converge.

Example. The limit supremum of the sequence

$\{t_n\} = \{4.1, 5.1, 4.01, 5.01, 4.001, 5.001, ...\}$ is $\limsup_{n\to\infty} t_n = 5$

, because if we set $L = 5$, then for any $\epsilon > 0$, there are only finitely many terms in the sequence larger than $L + \epsilon = 5 + \epsilon$. Additionally, if L is smaller than 5, then by setting $\epsilon = 5 - L$, we can find infinitely many terms in the sequence larger than $L + \epsilon$ (because $L + \epsilon = 5$).

Solution

In this case the even terms $t_{2k} = 4 + 10^{-1-k}$ tend to the limit 4 and the odd terms $t_{2k+1} = 5 + 10^{-1-k}$ tend to the limit 5.

The limit superior is the largest limit point of a subsequence of $\{t_n\}$.

We see that the limit supremum of the sequence $\{t_n\} = \{4.1, 5.1, 4.01, 5.01, 4.001, 5.001, ...\}$ is $\limsup_{n\to\infty} t_n = 5$.

Example. The limit supremum of the sequence $\{t_n\} = \{1, 2, 3, 1, 2, 3, 1, 2, 3, ...\}$ is $\limsup_{n\to\infty} t_n = 3$

, because if we set $L = 3$, then for any $\epsilon > 0$, there are only finitely many terms (actually, there are none) in the sequence larger than $L + \epsilon = 3 + \epsilon$. Additionally, if L is smaller than 3, then by setting $\epsilon = \frac{3 - L}{2}$ we can find infinitely many terms in the sequence larger than $L + \epsilon$, because $L + \epsilon < 3$, as the following calculation shows:

$$L + \epsilon = L + \frac{3 - L}{2} = \frac{3 + L}{2} = \frac{3}{2} + \frac{L}{2} < \frac{3}{2} + \frac{3}{2} = 3.$$

Solution

In this case there are only three different values for the terms in the sequence.

$$\{t_n\} = \{1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, \ldots\}$$

$$\limsup_{n\to\infty} t_n = \mathrm{Max}[1,2,3]$$

$$\limsup_{n\to\infty} t_n = 3$$

The limit superior is the largest limit point of a subsequence of $\{t_n\}$.

We see that the limit supremum of the sequence $\{t_n\} = \{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}$ is $\limsup_{n\to\infty} t_n = 3$.

Example. The limit supremum of the Fibonacci sequence $\{t_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ is $\limsup_{n\to\infty} t_n = \infty$. (The Fibonacci sequence satisfies the relation $t_n = t_{n-1} + t_{n-2}$ for $n > 2$).

Solution.

In this case the sequence has ∞ as its limit, and hence the limit supremum is also ∞.

$$\{t_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\}$$

$$t_n = -\frac{\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^{1+n}}{\sqrt{5}} + \frac{\left(\frac{2}{1+\sqrt{5}}\right)^{-1-n}}{\sqrt{5}}$$

$$\lim_{n\to\infty} t_n = \infty$$

$$\limsup_{n\to\infty} t_n = \infty$$

We see that the limit supremum of the Fibonacci sequence $\{t_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ is $\limsup_{n\to\infty} t_n = \infty$.

Example. The sequence $\{t_n\} = \left\{1 + \frac{1}{n}\right\} = \{2, 1.5, 1.3\overline{3}, 1.25, 1.2, \ldots\}$ has $\limsup_{n\to\infty} t_n = 1$. We leave verification of this as an exercise.

In this case the sequence has 1 as its limit, and hence the limit supremum is also 1.

Notes

6.6 ROOT TEST

In mathematics, the **root test** is a criterion for
the convergence (a convergence test) of an infinite series. It depends on the quantity where are the terms of the series, and states that the series converges absolutely if this quantity is less than one but diverges if it is greater than one. It is particularly useful in connection with power series.

Theorem (Root Test). Suppose that the series $\sum_{n=0}^{\infty} \xi_n$, has $\limsup_{n\to\infty} \left|\xi_n\right|^{\frac{1}{n}} = L$ (i.e. $\limsup_{n\to\infty} \sqrt[n]{|\xi_n|} = L$). Then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$.

6.7 SUMMARY

We study in this unit about root test, d'Alembert Ratio Test, We study in this unit Cauchy General Principal of Convergence with its examples. We study necessary condition for convergence.

6.8 KEYWORD

Root : The part of a plant which attaches it to the ground or to a support, typically underground, conveying water and nourishment to the rest of the plant via numerous branches and fibres

Finonacci : The Fibonacci sequence is a set of numbers that starts with a one or a zero, followed by a one, and proceeds based on the rule that each number (called a Fibonacci number) is equal to the sum of the preceding two numbers

Supremum : The *supremum* (abbreviated sup; plural suprema) of a subset S of a partially ordered set T is the least element in T that is greater than or equal to all elements of S, if such an element exists

6.9 QUESTIONS FOR REVIEW

Q. 1 The limit supremum of the Fibonacci sequence $\{t_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ is $\limsup_{n\to\infty} t_n = \infty$. (The Fibonacci sequence satisfies the relation $t_n = t_{n-1} + t_{n-2}$ for $n > 2$).

Q. 2 If $z \neq 1$, then for all n, $\dfrac{1}{1-z} = 1 + z + z^2 + \ldots + z^{n-1} + \dfrac{z^n}{1-z}$.

Q. 3 Show that $\sum_{n=0}^{\infty} \dfrac{(1-i)^n}{2^n} = 1 - i$.

Solution. If we set $z = \dfrac{1-i}{2}$, then $|z| = \left|\dfrac{1-i}{2}\right| = \dfrac{\sqrt{2}}{2} < 1$.

$$1 \Big/ \left(1 - \frac{1-i}{2}\right) = \frac{2}{2-1+i} = \frac{2}{1+i} = 1 - i.$$

Q. 4 If $|z| > 1$, the series $\sum_{n=1}^{\infty} z^{-n}$ converges to $f(z) = \dfrac{1}{z-1}$. That is,

if $|z| > 1$ then $\sum_{n=1}^{\infty} z^{-n} = z^{-1} + z^{-2} + \ldots + z^{-4} + \ldots = \dfrac{1}{z-1}$, or

equivalently,

6.10 SUGGESTION READING AND REFERENCES

[1] V.A. Il'in, E.G. Poznyak, "Fundamentals of mathematical analysis" , **1** , MIR (1982) (Translated from Russian)

[2] B.V. Shabat, "Introduction of complex analysis" , **1–2** , Moscow (1976) (In Russian)

[3] A.V. Bitsadze, "Fundamentals of the theory of analytic functions of a complex variable" , Moscow (1969) (In Russian) Zbl 0183.33601

6.11 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1

2 Check in Section 2

Check In Progress-II

Answer Q. 1 Check in section 3

2 Check in Section 4

UNIT 7: CONVERGENCE OF INFINITE PRODUCT

STRUCTURE

7.0 OBJECTIVE

- Learn infinite product of convergence series
- Learn Uniform convergence
- We study Taylor series representation
- We study Uniqueness of Power series

7.1 INTRODUCTION

In mathematics, for a sequence of complex numbers a_1, a_2, a_3,... the **infinite product** is defined to be the limit of the partial products $a_1a_2...a_n$ as n increases without bound. The product is said to *converge* when the limit exists and is not zero. Otherwise, the product is said to *diverge*. A limit of zero is treated specially in order to obtain results analogous to those for infinite sums. Some sources allow convergence to 0 if there are only a finite number of zero factors and the product of the non-zero factors is non-zero, but for simplicity we will not allow that here. If the product converges, then the limit of the sequence a_n as n increases without bound must be 1, while the converse is in general not true.

7.1.1 Convergence of Infinite Products

There is a simple convergence test for infinite products that I think deserves to be better known.

Theorem. *Let* a_n *be a sequence of positive numbers. Then the infinite product*

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if and only if the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

Proof: Taking the logarithm of the product gives the series

$$\sum_{n=1}^{\infty} \ln(1+a_n),$$

whose convergence is equivalent to the convergence of the product. But observe that

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1.$$

If we assume that $a_n \to 0$, this gives us that

$$\lim_{n\to\infty}\frac{\ln(1+a_n)}{a_n}=1,$$

and the theorem follows by the limit comparison test. Q.E.D.

Using this theorem, everything you know about infinite series translates directly to the world of infinite products. For example, the product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^p}\right)$$

converges if and only if $p > 1$.

Before I learned this theorem, I had imagined that there must be an entire theory of convergence for infinite products, as complex and interesting as the theory of series from calculus, but completely unknown to me. Instead, it turns out that no one ever talks about the convergence of infinite products because there is basically nothing new to say!

The Harmonic Series Another reason I like this theorem is that it gives a nice proof that the harmonic series diverges. According to the theorem, the behavior of the harmonic series is the same as the behavior of the following product:

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots$$

But this is just

$$\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\cdots$$

This clearly diverges, for the partial products are the sequence of positive integers.

Problems Finally, here's a fun little pair of exercises:

1. Find a sequence a_n of real numbers such that $\sum a_n$ converges but $\prod(1+a_n)$ diverges.

2. Find a sequence a_n of real numbers such that $\sum a_n$ diverges but $\prod(1+a_n)$ converges (and is greater than zero).

7.2 INFINITE PRODUCT

A product involving an infinite number of terms. Such products can converge. In fact, for positive a_n, the product $\prod_{n=1}^{\infty} a_n$ converges to a nonzero number iff $\sum_{n=1}^{\infty} \ln a_n$ converges.

Infinite products can be used to define the cosine

$$\cos x = \prod_{n=1}^{\infty}\left[1 - \frac{4x^2}{\pi^2 (2n-1)^2}\right], \tag{1}$$

gamma function

$$\Gamma(z) = \left[z\, e^{\gamma z} \prod_{r=1}^{\infty}\left(1 + \frac{z}{r}\right) e^{-z/r}\right]^{-1}, \tag{2}$$

sine, and sinc function. They also appear in polygon circumscribing,

$$K = \prod_{n=3}^{\infty} \frac{1}{\cos\left(\frac{\pi}{n}\right)}. \tag{3}$$

An interesting infinite product formula due to Euler which relates π and the nth prime p_n is

$$\pi = \frac{2}{\prod_{n=1}^{\infty}\left[1 + \frac{\sin\left(\frac{1}{2}\pi p_n\right)}{p_n}\right]} \tag{4}$$

$$= \frac{2}{\prod_{n=2}^{\infty}\left[1 + \frac{(-1)^{(p_n-1)/2}}{p_n}\right]} \tag{5}$$

(Blatner 1997). Knar's formula gives a functional equation for the gamma function $\Gamma(x)$ in terms of the infinite product

$$\Gamma(1+v) = 2^{2v} \prod_{m=1}^{\infty}\left[\pi^{-1/2}\, \Gamma\left(\tfrac{1}{2} + 2^{-m} v\right)\right]. \tag{6}$$

A regularized product identity is given by

$$\infty! = \hat{\prod_{k=1}^{\infty}} k = \sqrt{2\pi} \tag{7}$$

(Muñoz Garcia and Pérez-Marco 2003, 2008).

Mellin's formula states

$$\prod_{n=0}^{\infty}\left(1+\frac{y}{n+x}\right)e^{-y/(n+x)}=\frac{e^{y\,\psi_0(x)}\,\Gamma(x)}{\Gamma(x+y)}, \tag{8}$$

where $\psi_0(x)$ is the digamma function and $\Gamma(x)$ is the gamma function.

The following class of products

$$\prod_{n=2}^{\infty}\frac{n^2-1}{n^2+1}=\pi\operatorname{csch}\pi \tag{9}$$

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\tfrac{2}{3} \tag{10}$$

$$\prod_{n=2}^{\infty}\frac{n^4-1}{n^4+1}=-\tfrac{1}{2}\,\pi\sinh\pi\csc\left[(-1)^{1/4}\pi\right]\csc\left[(-1)^{3/4}\pi\right] \tag{11}$$

$$=\frac{\pi\sinh(\pi)}{\cosh\left(\sqrt{2}\,\pi\right)-\cos\left(\sqrt{2}\,\pi\right)} \tag{12}$$

$$\prod_{n=2}^{\infty}\frac{n^5-1}{n^5+1}=\frac{2\,\Gamma\left(-(-1)^{1/5}\right)\Gamma\left((-1)^{2/5}\right)\Gamma\left(-(-1)^{3/5}\right)\Gamma\left((-1)^{4/5}\right)}{5\,\Gamma\left((-1)^{1/5}\right)\Gamma\left(-(-1)^{2/5}\right)\Gamma\left((-1)^{3/5}\right)\Gamma\left(-(-1)^{4/5}\right)} \tag{13}$$

(Borwein *et al.* 2004, pp. 4-6), where $\Gamma(z)$ is the gamma function, the first of which is given in Borwein and Corless (1999), can be done analytically. In particular, for $r>1$,

$$\prod_{\substack{n=1\\ n\neq m}}^{\infty}\frac{n^r-m^r}{n^r+m^r}=(-1)^{m+1}\,\frac{2\,m\,m!}{r}\prod_{j=1}^{2r-1}\left[\Gamma\left(-m\,\omega_r^j\right)\right]^{(-1)^{j+1}}, \tag{14}$$

where $\omega_k=e^{i\pi/k}$ (Borwein *et al.* 2004, pp. 6-7). It is not known if (13) is algebraic, although it is known to satisfy no integer polynomial with degree less than 21 and Euclidean norm less than 5×10^{18} (Borwein *et al.* 2004, p. 7).

Products of the following form can be done analytically,

$$\prod_{k=1}^{\infty} \frac{(1+k^{-1})^2}{1+2\,k^{-1}} = 2$$

$$\prod_{k=1}^{\infty} \frac{(1+k^{-1}+k^{-2})^2}{1+2\,k^{-1}+3\,k^{-2}} = \frac{3\sqrt{2}\,\cosh^2\left(\frac{1}{2}\pi\sqrt{3}\right)\operatorname{csch}\left(\pi\sqrt{2}\right)}{\pi} \tag{15}$$

$$\prod_{k=1}^{\infty} \frac{(1+k^{-1}+k^{-2}+k^{-3})^2}{1+2\,k^{-1}+3\,k^{-2}+4\,k^{-3}} = \frac{\sinh^2\pi\prod_{i=1}^{3}\Gamma(x_i)}{\pi^2}$$

$$\prod_{k=1}^{\infty} \frac{(1+k^{-1}+k^{-2}+k^{-3}+k^{-4})^2}{1+2\,k^{-1}+3\,k^{-2}+4\,k^{-3}+5\,k^{-4}} = \prod_{i=1}^{4}\frac{\Gamma(y_i)}{\Gamma^2(z_i)},$$

where x_i, y_i, and z_i are the roots of

$$x^3 - 5\,x^2 + 10\,x - 10 = 0 \tag{16}$$

$$y^4 - 6\,y^3 + 15\,y^2 - 20\,y + 15 = 0 \tag{17}$$

$$z^4 - 5\,z^3 + 10\,z^2 - 10\,z + 5 = 0, \tag{18}$$

respectively, can also be done analytically. Note that (17) and (18) were unknown to Borwein and Corless (1999). These are special cases of the result that

$$\prod_{k=1}^{\infty} \frac{\sum_{i=1}^{p}\frac{a_i}{k^i}}{\sum_{i=0}^{q}\frac{b_i}{k^i}} = \frac{b_q}{a_p}\,\frac{\prod_{i=0}^{q}\Gamma(-s_i)}{\prod_{i=0}^{p}\Gamma(-r_i)}, \tag{19}$$

if $a_0 = b_0 = 1$ and $a_1 = b_1$, where r_i is the ith root of $\sum_{j=0}^{p} a_j / k^j$ and s_i is the ith root of $\sum_{j=0}^{q} b_j / k^j$ (P. Abbott, pers. comm., Mar. 30, 2006).

For $k \geq 2$,

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^k}\right) = \begin{cases} \dfrac{1}{k\prod_{j=1}^{k-1}\Gamma\left((-1)^{1+j\,(1+1/k)}\right)} & \text{for } k \text{ odd} \\[2ex] \dfrac{\prod_{j=1}^{(k/2)-1}\sin\left[\pi\,(-1)^{2\,j/k}\right]}{k\,(\pi\,i)^{(k/2)-1}} & \text{for } k \text{ even} \end{cases} \tag{20}$$

(D. W. Cantrell, pers. comm., Apr. 18, 2006). The first few explicit cases are

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2} \tag{21}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^3}\right)=\frac{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}{3\pi} \tag{22}$$

$$=\frac{1}{3\,\Gamma\left((-1)^{1/3}\right)\Gamma\left(-(-1)^{2/3}\right)} \tag{23}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^4}\right)=\frac{\sinh\pi}{4\pi} \tag{24}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^5}\right)=\frac{1}{5\,\Gamma\left((-1)^{1/5}\right)\Gamma\left(-(-1)^{2/5}\right)\Gamma\left((-1)^{3/5}\right)\Gamma\left(-(-1)^{4/5}\right)} \tag{25}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^6}\right)=\frac{1+\cosh\left(\pi\sqrt{3}\right)}{12\pi^2}. \tag{26}$$

These are a special case of the general formula

$$\prod_{k=1}^{\infty}\left(1-\frac{x^n}{k^n}\right)=-\frac{1}{x^n}\prod_{k=0}^{n-1}\frac{1}{\Gamma\left(-e^{2\pi i k/n}x\right)} \tag{27}$$

(Prudnikov *et al.* 1986, p. 754).

Similarly, for $k \geq 2$,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^k}\right)=\begin{cases}\dfrac{1}{\prod_{j=1}^{k-1}\Gamma\left[(-1)^{j(1+1/k)}\right]} & \text{for } k \text{ odd}\\[2ex] \dfrac{\prod_{j=1}^{k/2}\sin\left[\pi(-1)^{(2j-1)/k}\right]}{(\pi i)^{k/2}} & \text{for } k \text{ even}\end{cases} \tag{28}$$

(D. W. Cantrell, pers. comm., Mar. 29, 2006). The first few explicit cases are

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)=\frac{\sinh\pi}{\pi} \tag{29}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^3}\right)=\frac{1}{\pi}\cosh\left(\frac{1}{2}\pi\sqrt{3}\right) \tag{30}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^4}\right)=\frac{\cosh\left(\pi\sqrt{2}\right)-\cos\left(\pi\sqrt{2}\right)}{2\pi^2} \tag{31}$$

$$=-\frac{\sin\left[(-1)^{1/4}\pi\right]\sin\left[(-1)^{3/4}\pi\right]}{\pi^2} \tag{32}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^5}\right)=\left|\Gamma\left[\exp\left(\frac{2}{5}\pi i\right)\right]\Gamma\left[\exp\left(\frac{6}{5}\pi i\right)\right]\right|^{-2} \tag{33}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^6}\right)=\frac{\sinh\pi\left[\cosh\pi-\cos\left(\sqrt{3}\,\pi\right)\right]}{2\pi^3}. \tag{34}$$

The *d*-analog expression

$$[\infty!]_d = \prod_{n=3}^{\infty}\left(1 - \frac{2^d}{n^d}\right) \tag{35}$$

also has closed form expressions,

$$\prod_{n=3}^{\infty}\left(1 - \frac{4}{n^2}\right) = \frac{1}{6} \tag{36}$$

$$\prod_{n=3}^{\infty}\left(1 - \frac{8}{n^3}\right) = \frac{\sinh\left(\pi\sqrt{3}\right)}{42\,\pi\sqrt{3}} \tag{37}$$

$$\prod_{n=3}^{\infty}\left(1 - \frac{16}{n^4}\right) = \frac{\sinh(2\,\pi)}{120\,\pi} \tag{38}$$

$$\prod_{n=3}^{\infty}\left(1 - \frac{32}{n^5}\right) = \left|\Gamma\left[\exp\left(\tfrac{1}{5}\,\pi\,i\right)\right]\Gamma\left[2\exp\left(\tfrac{7}{5}\,\pi\,i\right)\right]\right|^{-2}. \tag{39}$$

General expressions for infinite products of this type include

$$\prod_{n=1}^{\infty}\left[1 - \left(\frac{z}{n}\right)^{2N}\right] = \frac{\sin(\pi z)}{\pi\,z^{2N-1}}\prod_{k=1}^{N-1}\left|\Gamma\left(z\,e^{2\pi i(k-N)/(2N)}\right)\right|^{-2} \tag{40}$$

$$\prod_{n=1}^{\infty}\left[1 + \left(\frac{z}{n}\right)^{2N}\right] = \frac{1}{z^{2N}}\prod_{k=1}^{N}\left|\Gamma\left(z\,e^{\pi i[2(k-N)-1]/(2N)}\right)\right|^{-2} \tag{41}$$

$$\prod_{n=1}^{\infty}\left[1 - \left(\frac{z}{n}\right)^{2N+1}\right] = \frac{1}{\Gamma(1-z)\,z^{2N}}\prod_{k=1}^{N}\left|\Gamma\left(z\,e^{\pi i[2(k-N)-1]/(2N+1)}\right)\right|^{-2} \tag{42}$$

$$\prod_{n=1}^{\infty}\left[1 + \left(\frac{z}{n}\right)^{2N+1}\right] = \frac{1}{\Gamma(1+z)\,z^{2N}}\prod_{k=1}^{N}\left|\Gamma\left(z\,e^{2\pi i(k-N-1)/(2N+1)}\right)\right|^{-2}, \tag{43}$$

where $\Gamma(z)$ is the gamma function and $|z|$ denotes the complex modulus (Kahovec). (40) and (41) can also be rewritten as

$$\prod_{n=1}^{\infty}\left[1 - \left(\frac{z}{n}\right. = \frac{\sin(\pi z)}{\pi^3 z^2}\left[\frac{\sinh(\pi z)}{\pi z}\right]^{\mathrm{mod}(N+1,2)} \times \prod_{k=1}^{\lceil N/2\rceil-1}\cosh^2\left[\pi z\sin\left(\frac{k\pi}{N}\right)\right] - \cos^2 \tag{44}$$

$$\prod_{n=1}^{\infty}\left[1 + \left(\frac{z}{n}\right. = \frac{1}{\pi^2 z^2}\left[\frac{\sinh(\pi z)}{\pi z}\right]^{\mathrm{mod}(N,2)} \times \prod_{k=1}^{\lfloor N/2\rfloor}\cosh^2\left[\pi z\sin\left(\frac{(2k-1)\pi}{2N}\right)\right] - \cos^2 \tag{45}$$

where $\lfloor x\rfloor$ is the floor function, $\lceil x\rceil$ is the ceiling function, and $\mathrm{mod}(a, m)$ is the modulus of a (mod m) (Kahovec).

Infinite products of the form

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{n^k}\right) = \left(n^{-1}\right)_\infty \tag{46}$$

$$= n^{1/24}\left[\tfrac{1}{2}\,\vartheta_1'\left(0,\, n^{-1/2}\right)\right]^{1/3} \tag{47}$$

converge for $n > 1$, where $(q)_\infty$ is a *q*-Pochhammer symbol and $\vartheta_n(z, q)$ is a Jacobi theta function. Here, the $n = 2$ case is exactly the constant Q encountered in the analysis of digital tree searching.

Other products include

$$\prod_{k=1}^{\infty}\left(1 + \frac{2}{k}\right)^{(-1)^{k+1}\,k} = \frac{\pi}{2\,e} \tag{48}$$

$$= 0.57786367\ldots \tag{49}$$

$$\prod_{k=0}^{\infty}\left(1 + e^{-(2k+1)\pi}\right) = 2^{1/4}\,e^{-\pi/24} \tag{50}$$

$$\prod_{k=3}^{\infty}\left(1 - \frac{\pi^2}{2\,k^2}\right)\sec\left(\frac{\pi}{k}\right) = 0.86885742\ldots \tag{51}$$

(OEIS A086056 and A247559; Prudnikov *et al.* 1986, p. 757). Note that Prudnikov *et al.* (1986, p. 757) also incorrectly give the product

$$\prod_{k=1}^{\infty}\left(1 - e^{-2\pi k/\sqrt{3}}\right) = \left(e^{-2\pi/\sqrt{3}}\right)_\infty, \tag{52}$$

where $(q)_\infty$ is a *q*-Pochhammer symbol, as $3^{1/4}\,e^{-\pi/\left(6\sqrt{3}\right)}$, which differs from the correct result by 1.8×10^{-5}.

The following analogous classes of products can also be done analytically (J. Zúñiga, pers. comm., Nov. 9, 2004), where again $\vartheta_n(z, q)$ is a Jacobi theta function,

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{n^k}\right) = n^{1/24}\,\vartheta_4^{-1/2}\left(0,\, n^{-1}\right)\left[\tfrac{1}{2}\,\vartheta_1'\left(0,\, n^{-1}\right)\right]^{1/6} \tag{53}$$

$$\prod_{k=1}^{\infty}\left(\frac{1 - n^{-k}}{1 + n^{-k}}\right) = \prod_{k=1}^{\infty}\tanh\left(\tfrac{1}{2}\,k\ln n\right) \tag{54}$$

$$= \vartheta_4\left(0,\, n^{-1}\right) \tag{55}$$

$$\prod_{k=1}^{\infty}\left(\frac{1 - n^{-2k}}{1 + n^{-2k}}\right)^2 = \prod_{k=1}^{\infty}\tanh^2(k\ln n) \tag{56}$$

$$= \frac{\vartheta_1'\left(0,\, n^{-1}\right)}{\vartheta_2\left(0,\, n^{-1}\right)} \tag{57}$$

$$\prod_{k=1}^{\infty}\left(\frac{1 - n^{-2k+1}}{1 + n^{-2k+1}}\right)^2 = \prod_{k=1}^{\infty}\tanh^2\left[\left(k - \tfrac{1}{2}\right)\ln n\right] \tag{58}$$

$$= \frac{\vartheta_4\left(0,\, n^{-1}\right)}{\vartheta_3\left(0,\, n^{-1}\right)} \tag{59}$$

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{n^{2k-1}}\right) = n^{-1/24}\,\vartheta_4^{1/2}\left(0,\, n^{-1}\right)\left[\frac{2}{\vartheta_1'\left(0,\, n^{-1}\right)}\right]^{1/6} \tag{60}$$

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{n^{2k-1}}\right) = n^{-1/24}\,\vartheta_3^{1/2}\left(0,\, n^{-1}\right)\left[\frac{2}{\vartheta_1'\left(0,\, n^{-1}\right)}\right]^{1/6} \tag{61}$$

$$\prod_{k=1}^{\infty}\left[1 + (-1)^{k-1}\,\frac{b}{k+a}\right] = 2^b\,{}_2F_1\,(a+b,\, b;\, a+1;\, -1) \tag{62}$$

$$= \frac{\sqrt{\pi}\;\Gamma(a+1)}{2^a\,\Gamma\left(\frac{1}{2}\,(2+b-a)\right)\Gamma\left(\frac{1}{2}\,(1+b+a)\right)}. \tag{63}$$

The first of these can be used to express the Fibonacci factorial constant in closed form.

A class of infinite products derived from the Barnes G-function is given by

$$\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)^n e^{-z+z^2/(2n)} = \frac{G(z+1)}{(2\pi)^{z/2}}\, e^{[z(z+1)+\gamma z^2]/2}, \tag{64}$$

where γ is the Euler-Mascheroni constant. For $z = 1$, 2, 3, and 4, the explicit products are given by

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)^n e^{1/(2n)-1} = \frac{e^{1+\gamma/2}}{\sqrt{2\pi}} \tag{65}$$

$$\prod_{n=1}^{\infty}\left(1 + \frac{2}{n}\right)^n e^{4/(2n)-2} = \frac{e^{3+2\gamma}}{2\pi} \tag{66}$$

$$\prod_{n=1}^{\infty}\left(1 + \frac{3}{n}\right)^n e^{9/(2n)-3} = \frac{e^{6+9\gamma/2}}{\sqrt{2}\;\pi^{3/2}} \tag{67}$$

$$\prod_{n=1}^{\infty}\left(1 + \frac{4}{n}\right)^n e^{16/(2n)-4} = \frac{3\,e^{10+8\gamma}}{\pi^2}. \tag{68}$$

The interesting identities

$$x\prod_{n=1}^{\infty}\frac{\left(1 - x^{2n}\right)^8}{\left(1 - x^{2n-1}\right)^8} = \sum_{n=1}^{\infty} 2^{3\,b(n)}\,\sigma_3\,(\mathrm{Od}\,(n))\,x^n \tag{69}$$

(Ewell 1995, 2000), where $b(n)$ is the exponent of the exact power of 2 dividing n, $\mathrm{Od}\,(n) = n/2^{b(n)}$ is the odd part of n, $\sigma_k(n)$ is the divisor function of n, and

Notes

$$\prod_{n=1}^{\infty}\left(1+x^{2n-1}\right)^8=\prod_{n=1}^{\infty}\left(1-x^{2n-1}\right)^8+16x\prod_{n=1}^{\infty}\left(1+x^{2n}\right)^8 \quad (70)$$

$$=1+8x+28x^2+64x^3+134x^4+288x^5+\ldots \quad (71)$$

(OEIS A101127; Jacobi 1829; Ford *et al.* 1994; Ewell 1998, 2000), the latter of which is known as "aequatio identica satis abstrusa" in the string theory physics literature, arise in connection with the tau function.

An unexpected infinite product involving $\tan x$ is given by

$$\left|\prod_{k=0}^{\infty}\left[\tan\left(2^k x\right)\right]^{1/\left(2^k\right)}\right|=4\sin^2 x \quad (72)$$

(Dobinski 1876, Agnew and Walker 1947).

A curious identity first noted by Gosper is given by

$$\prod_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{3n}+1\right)^{3n+1/2}=\sqrt{\frac{\Gamma\left(\frac{1}{3}\right)}{2\pi}}\;\frac{3^{13/24}\exp\left[1+\frac{2\pi^2-3\psi_1\left(\frac{1}{3}\right)}{12\pi\sqrt{3}}\right]}{A^4} \quad (73)$$

$$=1.012378552722912\ldots \quad (74)$$

7.3 UNIFORM CONVERGENCE

Complex functions are the key to unlocking many of the mysteries encountered when power series are first introduced in a calculus course. We begin by discussing an important property associated with power series-uniform convergence.

Recall that, for a function f(z) defined on a set T, the sequence of functions $\{S_n(z)\}$ converges to the function f(z) at the point $z_0 \in T$ provided that $\lim_{n\to\infty} S(z_0) = f(z_0)$. Thus, for the particular point z_0, we know that for each $\epsilon > 0$, there exists a positive integer N_{ϵ,z_0} (which depends on both ϵ and z_0) such that if $n \geq N_{\epsilon,z_0}$, then $|S(z_0) - f(z_0)| < \epsilon$. If $S_n(z)$ is the n^{th} partial sum of the series

$\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, Statement (7-1) becomes If $n \geq N_{\epsilon, z_0}$, then

$$\left| \sum_{k=0}^{n-1} c_k (z_0-\alpha)^k - f(z_0) \right| < \epsilon.$$

For a given value of ϵ, the integer N_{ϵ, z_0} needed to satisfy Statement (7-1) often depends on our choice of z_0. This is not the case if the sequence $\{S_n(z)\}$ converges uniformly. For a uniformly convergent sequence, it is possible to find an integer N_ϵ (depending only on ϵ) that guarantees Statement 1 no matter what value for $z_0 \in T$ we pick. In other words, if n is large enough, the function $S_n(z)$ is uniformly close to the function f(z) for all $z \in T$. Formally, we have the following definition.

Definition 1 (Uniform Convergence),. The sequence $\{S_n(z)\}$ converges uniformly to f(z) on the set T if for every $\epsilon > 0$, there exists a positive integer N_ϵ (depending only on ϵ) such that (7-2) if $n \geq N$, then $|S_n(z) - f(z)| < \epsilon$ for all $z \in T$. If $S_n(z)$ is the n^{th} partial sum of the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, we say that the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges uniformly to f(z) on the set T.

A sequence of functions $\{f_n\}$, $n=1$, 2, 3,... is said to be uniformly convergent to f for a set E of values of x if, for each $\epsilon > 0$, an integer N can be found such that

$$|f_n(x) - f(x)| < \epsilon \qquad (1)$$

for $n \geq N$ and all $x \in E$.

A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{k=1}^{n} f_k(x) = S_n(x) \qquad (2)$$

converges uniformly on E.

To test for uniform convergence, use Abel's uniform convergence test or the Weierstrass M-test. If individual terms $u_n(x)$ of a uniformly converging series are continuous, then the following conditions are satisfied.

1. The series sum

$$f(x) = \sum_{n=1}^{\infty} u_n(x) \tag{3}$$

is continuous.

2. The series may be integrated term by term

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx. \tag{4}$$

For example, a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is uniformly convergent on any closed and bounded subset inside its circle of convergence.

3. The situation is more complicated for differentiation since uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ does not tell anything about convergence of $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$. Suppose that $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some $x_0 \in [a, b]$, that each $u_n(x)$ is differentiable on $[a, b]$, and that $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$ converges uniformly on $[a, b]$. Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b]$ to a function f, and for each $x \in [a, b]$,

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x).$$

Example. The sequence $\{S_n(z)\} = \left\{e^z + \frac{1}{n}\right\}$ converges uniformly to the function $f(z) = e^z$ on the entire complex plane because for any $\varepsilon > 0$, statement (7-2) is satisfied for all z for $n \geq N_\varepsilon$, where N_ε is any integer greater than $\frac{1}{\varepsilon}$. We leave the details of showing this result as an exercise.

A good example of a sequence of functions that does not converge uniformly is the sequence of partial sums comprising the geometric series. Recall that the geometric series has $S_n(z) = \sum_{k=0}^{n} z^k$ converging to $f(z) = \frac{1}{1-z}$ for all $z \in D_1(0)$. Because the real numbers are a subset of the complex numbers, we can show statement (1) is not satisfied by demonstrating it does not hold when we restrict our attention to the real numbers. In that context, $D_1(0)$ becomes the open interval (-1,1), and the inequality, $|S_n(z) - f(z)| < \epsilon$, becomes $|S_n(x) - f(x)| < \epsilon$, which for real variables is equivalent to $f(x) - \epsilon < S_n(x) < f(x) + \epsilon$. If Statement (1) were to be satisfied, then given $\epsilon > 0$, $S_n(x)$ would be within an ϵ-bandwidth of f(x) for all x in the interval (-1,1) provided n was large enough. This illustrates that there is a such that, no matter how large n is, we can find $x_0 \in (-1, 1)$ such that $S_n(x_0)$ lies outside this bandwidth. In other words, illustrates the negation of e which in technical terms we state as: There exists $\epsilon > 0$, such that for all positive integers N, there is some $n \geq N$ and some $z_0 \in T$ such that $|S_n(z_0) - f(z_0)| \geq \epsilon$. In the exercises, we ask you to use Statement (7-3) to show that the partial sums of the geometric series do not converge uniformly to $f(z) = \frac{1}{1-z}$ for $z \in D_1(0)$.

A useful procedure known as the Weierstrass M-test can help determine whether an infinite series is uniformly convergent.

7.4 WEIERSTRASS M-TEST

Theorem A (Weierstrass M-Test) Suppose the infinite series $\sum_{k=0}^{\infty} u_k(z)$ has the property that for each k, we have $|u_k(z)| \leq M_k$ for all $z \in T$. If $\sum_{k=0}^{\infty} M_k$ converges, then $\sum_{k=0}^{\infty} u_k(z)$ converges uniformly on T.

Proof.

Theorem A gives an interesting application of the Weierstrass M-test.

Theorem B. Suppose the power series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ has radius of convergence $\rho > 0$. Then for each r, (where $0 < r < \rho$), the series converges uniformly on the closed disk $\overline{D_r}(\alpha) = \{z : |z-\alpha| \le r\}$.

Corollary 1. For each r, (where $0 < r < 1$), the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly on the closed disk $\overline{D_r}(0) = \{z : |z| \le r\}$.

Theorem C. Suppose $\{S_k(z)\}$ is a sequence of continuous functions defined on a set T containing the contour C. If $\{S_k(z)\}$ converges uniformly to $f(z)$ on the set T, then (i) $f(z)$ is continuous on T, and (ii) $\lim_{k\to\infty} \int_C S_k(z)\,dz = \int_C \lim_{k\to\infty} S_k(z)\,dz = \int_C f(z)\,dz$.

Corollary 2. If the series $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ converges uniformly to $f(z)$ on the set T, and C is a contour contained in T, then

$$\sum_{n=0}^{\infty} \int_C c_n (z-\alpha)^n\,dz = \int_C \sum_{n=0}^{\infty} c_n (z-\alpha)^n\,dz = \int_C f(z)\,dz.$$

Example Show that $-\mathrm{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ for all $z \in D_1(0) = \{z : |z| < 1\}$.

Solution. For $z_0 \in D_1(0)$, we choose r and R so that $0 \le |z_0| < r < R < 1$, thus ensuring that $z_0 \in \overline{D_r}(0)$ and that $\overline{D_r}(0) \subset D_R(0)$. By Corollary 7.1, the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly to $f(z) = \frac{1}{1-z}$ on $\overline{D_r}(0)$. If C is any contour contained in $\overline{D_r}(0)$, Corollary 7.2 gives (7-4) $\int_C \frac{1}{1-z}\,dz = \sum_{n=0}^{\infty} \int_C z^n\,dz$.

Clearly, the function $f(z) = \frac{1}{1-z}$ is analytic in the simply connected domain $D_R(0)$, and $F(z) = -\mathrm{Log}(1-z)$ is an antiderivative of f(z) for all $z \in D_R(0)$, where Log is the principal branch of the logarithm. Likewise, $g(z) = z^n$ is analytic in the simply connected domain $D_R(0)$, and $G(z) = \frac{1}{n+1} z^{n+1}$ is an antiderivative of g(z) for all $z \in D_R(0)$.

Hence, if C is the straight-line segment joining 0 to z_0, we can apply Theorem 6.9 to Equation (7-4) to get

$$-\text{Log}(1-z) \Big|_{z=0}^{z=z_0} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} z^{n+1} \Big|_{z=0}^{z=z_0} \right)$$, which becomes

$$-\text{Log}(1-z_0) + \text{Log}(1) = \sum_{n=0}^{\infty} \frac{1}{n+1} z_0^{n+1}$$, which can be written as

$$-\text{Log}(1-z_0) = \sum_{n=1}^{\infty} \frac{1}{n} z_0^n$$. The point $z_0 \in D_1(0)$ was arbitrary, so we are done.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 State Weierstrass M-Test.

Solution :

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Q. 2 State Uniform Convergence.

Solution :

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7.5 TAYLOR SERIES REPRESENTATIONS

we showed that functions defined by power series have derivatives of all orders we demonstrated that analytic functions also have derivatives of all orders. It seems natural, therefore, that there would be some connection between analytic functions and power series. As you might guess, the connection exists via the Taylor and Malaren series of analytic functions.

Notes

Definition (Taylor Series). If $f(z)$ is analytic at $z=\alpha$, then the series

$$f(\alpha)+f'(\alpha)(z-\alpha)+\frac{f^{(2)}(\alpha)}{2!}(z-\alpha)^2+\frac{f^{(3)}(\alpha)}{3!}(z-\alpha)^3+\ldots$$

$$=\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

is called the Taylor series for f(z) centered at $z=\alpha$. When the center is $\alpha=0$, the series is called the Maclaurin series for f(z).

To investigate when these series converge we will need the following lemma.

Lemma. If z, z_0 and α are complex numbers with $z \neq z_0$, and $z \neq \alpha$, then

$$\frac{1}{z-z_0}=\frac{1}{z-\alpha}+\frac{z_0-\alpha}{(z-\alpha)^2}+\frac{(z_0-\alpha)^2}{(z-\alpha)^3}+\ldots+\frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}}+\frac{1}{z-z_0}\frac{(z_0-\alpha)^{n+1}}{(z-\alpha)^{n+1}}$$

where n is a positive integer.

We are now ready for the main result of this section.

Theorem (Taylor's Theorem). Suppose f(z) is analytic in a domain G, and that $D_R(\alpha)=\{z: |z-\alpha|<R\}$ is any disk contained in G. Then the Taylor series for f(z) converges to f(z) for all z in $D_R(\alpha)$; that is,

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

for all $z \in D_R(\alpha)$. Furthermore, for any r, 0<r<R, the convergence is uniform on the closed subdisk $\overline{D_r}(\alpha)=\{z: |z-\alpha|\leq r\}$ for $0<r<R$.

singular point of a function is a point at which the function fails to be analytic. You will see in Section 7.4 that singular points of a function can be classified according to how badly the function behaves at those points. Loosely speaking, a nonremovable singular point of a function has the property that it is impossible to redefine the value of the function at that point so as to make it analytic there. For example, the function $f(z)=\frac{1}{1-z}$ has a nonremovable singularity at z=1. We give a formal definition of this concept in Section 7.4, but with this language, we can nuance Taylor's theorem a bit.

Corollary. Suppose that f(z) is analytic in the domain G that contains the point $z = \alpha$. Let z_0 be a nonremovable singular point of minimum distance to the point $z = \alpha$. If $|z_0 - \alpha| = R$, then (i) the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$ converges to $f(z)$ on all of $D_R(\alpha)$, and (ii) if $|z_1 - \alpha| = S > R$, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z_1-\alpha)^n$ does not converges to $f(z_1)$.

Example. Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n$ is valid for all $z \in D_1(0) = \{z : |z| < 1\}$.

Solution. we established this identity with the use of Theorem 4.17. We now do so. If $f(z) = \frac{1}{(1-z)^2}$, then a standard induction argument (which we leave as an exercise) will show that $f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}}$ for $z \in D_1(0)$. Thus $f^{(n)}(0) = (n+1)!$, and Taylor's theorem gives

$$f(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} z^n = \sum_{n=0}^{\infty}(n+1)z^n,$$

and since f(z) is analytic in $D_1(0)$, this series expansion is valid for all $z \in D_1(0)$.

The disk $D_{\frac{3}{5}}(0) = \left\{|z| < \frac{3}{5}\right\}$ and it's images under the mappings: $w = S_8(z) = \sum_{n=0}^{8}(n+1)z^n$, $w = S_{12}(z) = \sum_{n=0}^{12}(n+1)z^n$, and $w = S_{16}(z) = \sum_{n=0}^{16}(n+1)z^n$.

Remark. The accuracy of the image points for the approximation $w = f(z) = \frac{1}{(1-z)^2} \approx \sum_{n=0}^{16}(n+1)z^n$ is

$$|f(z) - s_{16}(z)| = \left|\sum_{n=17}^{\infty}(n+1)z^n\right| = \sum_{n=17}^{\infty}\left|(n+1)z^n\right| \le \sum_{n=17}^{\infty}(n+1)\left(\frac{3}{5}\right)^n \approx 0.0082517.$$

.

Example. Show that, for $z \in D_1(0)$, (a) $\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$ and (b) $\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$.

Solution. For $z \in D_1(0)$, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. If we let z^2 take the role of z in (7-13), we get that $\frac{1}{1-z^2} = \frac{1}{1-(z^2)} = \sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} z^{2n}$, for $z^2 \in D_1(0)$. But $z^2 \in D_1(0)$ iff $z \in D_1(0)$, thus we have proven that $\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$ for $z \in D_1(0)$. Next, let $-z^2$ take the role of z in Equation (7-13), we get that $\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ gives the second part of Equations (7-12).

The disk $D_{\frac{9}{10}}(0) = \left\{|z| < \frac{9}{10}\right\}$ and it's images under the mappings: $w = s_{12}(z) = \sum_{n=0}^{12} z^{2n}$, $w = s_{16}(z) = \sum_{n=0}^{16} z^{2n}$, and $w = s_{20}(z) = \sum_{n=0}^{20} z^{2n}$.

Remark. The accuracy of the image points for the approximation $w = f(z) = \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$ is $\left|f(z) - \sum_{n=0}^{20} z^{2n}\right| = \left|\sum_{n=21}^{\infty} z^{2n}\right| \le \sum_{n=21}^{\infty}\left(\frac{9}{10}\right)^{2n} \approx 0.0630132$. Remark 2. The images of $D_{\frac{9}{10}}(0)$ under the mappings: $w = s_{12}(z) = \sum_{n=0}^{12}(-1)^n z^{2n}$, $w = s_{16}(z) = \sum_{n=0}^{16}(-1)^n z^{2n}$, and $w = s_{20}(z) = \sum_{n=0}^{20}(-1)^n z^{2n}$ will appear like those shown above, because $i z$ rotates the plane about the origin and $(-1)^n z^{2n} = ((i)^2)^n z^{2n} = (i z)^{2n}$. Also, the accuracy of the image points for the approximation $w = f(z) = \frac{1}{1 - +z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$ will be

$$\left| f(z) - \sum_{n=0}^{20} (-1)^n z^{2n} \right| = \left| \sum_{n=21}^{\infty} (-1)^n z^{2n} \right| \le \sum_{n=21}^{\infty} \left| (-1)^n \left(\frac{9}{10}\right)^{2n} \right| = \sum_{n=21}^{\infty} \left(\frac{9}{10}\right)^{2n} \approx 0.0630132$$

Remark. This clears up what often seems to be a mystery when series are first introduced in calculus. The calculus analog is (7-14)

$\frac{1}{1 - x^2} = \sum_{n=0}^{\infty} x^{2n}$ and $\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $x \in (-1, 1)$. For many students, it makes sense that the first series in Equations (7-14) converges only on the interval $(-1, 1)$ because $\frac{1}{1 - x^2}$ is undefined at the points $x = \pm 1$. It seems unclear as to why this should also be the case for the series representing $\frac{1}{1 + x^2}$, since the real-valued function $f(x) = \frac{1}{1 + x^2}$ is defined everywhere. The explanation, of course, comes from the complex domain. The complex function $f(z) = \frac{1}{1 + z^2}$ is not defined everywhere. In fact, the singularities of $f(z)$ are at the points $z = \pm i$, and the distance between them and the point $z = \alpha = 0$ equals 1. therefore, Equations 3 are valid only for $z \in D_1(0)$, and thus Equations are valid only for the real numbers $x \in (-1, 1)$.

Alas, there is a potential fly in this ointment: applies to Taylor series. To form the Taylor series of a function, we must compute its derivatives. We didn't get the series in Equations by computing derivatives, so how do we know that they are indeed the Taylor series centered at $z = \alpha = 0$? Perhaps the Taylor series would give completely different expressions from those given Fortunately, removes this possibility.

Theorem (Uniqueness of Power Series). Suppose that in some disk $D_r(\alpha)$ we have $f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n = \sum_{n=0}^{\infty} b_n (z - \alpha)^n$. Then $a_n = b_n$ for $n = 0, 1, 2, \ldots$.

Notes

Example Find the Maclaurin series for $f(z) = \sin^3 z$.

Solution. Computing derivatives for f(z) would be an onerous task. Fortunately, we can make use of the trigonometric identity $\sin^3 z = \frac{3}{4}\sin z - \frac{1}{4}\sin 3z$. Recall that the series for sin z (valid for all z) is $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$. Using the identity for $\sin^3 z$, we obtain

$$\sin^3 z = \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (3z)^{2n+1}$$

$$= \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} - \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n+1)!} z^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3(1-9^n)}{4(2n+1)!} z^{2n+1}$$

By the uniqueness of power series, this last expression is the Maclaurin series for $\sin^3 z$.

In the preceding argument we used some obvious results of power series representations that we haven't yet formally stated.

Theorem. Let f(z) and g(z) have the power series representations

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \text{ for } z \in D_{r_1}(\alpha),$$

and $g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n$ for $z \in D_{r_2}(\alpha)$. If $r = \min\{r_1, r_2\}$ and β is any complex constant, then (7-15)

$$\beta f(z) = \sum_{n=0}^{\infty} \beta a_n (z-\alpha)^n \text{ for } z \in D_{r_1}(\alpha), \quad (7\text{-}16)$$

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z-\alpha)^n \text{ for } z \in D_r(\alpha), \text{ and}$$

$$f(z) * g(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \text{ for } z \in D_r(\alpha), \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Identity is known as the Cauchy product of the series for f(z) and g(z).

Example. Use the Cauchy product of series to show that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \text{ for } z \in D_1(0).$$

Solution. We let $f(z) = g(z) = \frac{1}{1-z}$, for $z \in D_1(0)$. we have

$$\frac{1}{(1-z)^2} = h(z) = f(z)\,g(z)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\, b_{n-k}\right) z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\right) z^n$$

$a_n = b_n = 1$, for all n, and thus

$$= \sum_{n=0}^{\infty} (n+1)\, z^n$$

Soloution :

Use the following fact that $f(z) = g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $z \in D_1(0)$.

Enter the coefficients $a_n = 1$ and $b_n = 1$ and let *Mathematica* carry out the computations.

$a_n = 1$

$b_n = 1$

$$f[z] = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (1)\, z^n = \frac{1}{1-z}$$

$$g[z] = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (1)\, z^n = \frac{1}{1-z}$$

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$$

$$c_n = \sum_{k=0}^{n} (1)(1) = \sum_{k=0}^{n} (1)$$

$$\sum_{k=0}^{n} (1) = 1 + n$$

$$c_n = 1 + n$$

$$h[z] = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} (1+n)\, z^n = \frac{1}{(-1+z)^2}$$

Extra Example 1. Use the Cauchy product of series to show that

$$\frac{z}{(z-1)^4} = \sum_{n=0}^{\infty} \frac{n(1+n)(2+n)}{6}\, z^n$$ for $z \in D_1(0)$.

Notes

Explore Solution for Extra Example 1.

Use the following facts $\frac{z}{(z-1)^2} = \sum_{n=0}^{\infty} n\, z^n \text{ for } z \in D_1(0)$ and $\frac{1}{(z-1)^2} = \sum_{n=0}^{\infty} (n+1)\, z^n \text{ for } z \in D_1(0)$. Enter the coefficients $a_n = n$ and $b_n = n+1$ and let *Mathematica* carry out the computations.

$$a_n = n$$
$$b_n = 1+n$$

$$f[z] = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (k)\, z^n = \frac{z}{(-1+z)^2}$$

$$g[z] = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (1-k+n)\, z^n = \frac{1}{(-1+z)^2}$$

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$$

$$c_n = \sum_{k=0}^{n} (k)\,(1-k+n) = \sum_{k=0}^{n} (k\,(1-k+n))$$

$$\sum_{k=0}^{n} (k\,(1-k+n)) = \frac{1}{6}\, n\,(1+n)\,(2+n)$$

$$c_n = \frac{1}{6}\, n\,(1+n)\,(2+n)$$

$$h[z] = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} \left(\frac{1}{6}\, n\,(1+n)\,(2+n)\right) z^n = \frac{z}{(-1+z)^4}$$

Extra Example 2. Show that $\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n\, z^n$ for $z \in D_1(0)$.

$$\frac{z}{(1-z)^2} = \frac{z-1+1}{(1-z)^2}$$

$$= \frac{1}{(1-z)^2} + \frac{-1}{(1-z)}$$

$$= \sum_{n=0}^{\infty} (n+1) z^n - \sum_{n=0}^{\infty} z^n$$

$$= \sum_{n=0}^{\infty} n z^n$$

Solution. Now we obtain $= \sum_{n=0}^{\infty} n z^n$

$$\sum_{n=1}^{\infty} n z^n = \frac{z}{(-1+z)^2}$$

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 Define Taylor Series.

Solution :

………………………………………………………………………………

……………………………………………………………………………….

………………………………………………………………………………

Q. 2 Use the Cauchy product of series to show that

$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$ for $z \in D_1(0)$.

Solution :

………………………………………………………………………………

………………………………………………………………………………

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Notes

7.6 EXERCISES FOR TAYLOR SERIES REPRESENTATIONS

Exercise 1. By computing derivatives, find the Maclaurin series for each function and state where it is valid.

1 (a). $\sinh(z)$.

Answer. $\sinh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}$ valid for all z.

Solution. Given $f(z) = \sinh(z)$, the derivatives are $f'(z) = \cosh(z)$, $f^{(2)}(z) = \sinh(z)$, $f^{(3)}(z) = \cosh(z)$, etc. In general the even derivatives are $f^{(2k)}(z) = \sinh(z)$ for $k = 0, 1, 2, \ldots$, and the odd derivatives are $f^{(2k+1)}(z) = \cosh(z)$ for $k = 0, 1, 2, \ldots$. Now evaluate these derivatives at $z = 0$ and get: $f^{(2k)}(0) = \sinh(0) = 0$ for $k = 0, 1, 2, \ldots$, $f^{(2k+1)}(0) = \cosh(0) = 1$ for $k = 0, 1, 2, \ldots$.

the coefficients of the Maclaurin series are $a_{2k} = \frac{f^{(2k)}(0)}{(2k)!} = 0$ for $k = 0, 1, 2, \ldots$, $a_{2k+1} = \frac{f^{(2k+1)}(0)}{(2k+1)!} = \frac{1}{(2k+1)!}$ for $k = 0, 1, 2, \ldots$. and the sequence of coefficients is

$$0, 1, 0, \frac{1}{3!}, 0, \frac{1}{5!}, 0, \frac{1}{7!}, 0, \frac{1}{9!}, 0, \frac{1}{11!}, 0, \frac{1}{13!}, \ldots$$

Or if you prefer, you can write it as:

$$a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{1}{n!}, & \text{for } n \text{ odd} \end{cases}$$

The series is usually expressed by adding up the non-zero odd powers

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \frac{z^9}{9!} + \frac{z^{11}}{11!} + \frac{z^{13}}{13!} + \ldots$$

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1}$$

Or if you prefer the series can be written as

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2} \frac{1}{n!} z^n$$

We are done.

Remark. If this last series looks strange, then recall

that $\sinh(z) = \frac{1}{2} e^{z} - \frac{1}{2} e^{-z}$ and

that $e^{-z} = \sum_{n=0}^{\infty} \frac{1}{n!} (-z)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} z^n$. Hence, we obtain

$$\sinh(z) = \frac{1}{2} e^{z} - \frac{1}{2} e^{-z}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \frac{1}{n!} z^n - \sum_{n=0}^{\infty} \frac{1}{2} (-1)^n \frac{1}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2} \frac{1}{n!} z^n$$

1 (b). $\cosh(z)$.

Answer. $\cosh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$ valid for all z.

Solution. Given $f(z) = \cosh(z)$, the derivatives are $f'(z) = \sinh(z)$, $f^{(2)}(z) = \cosh(z)$, $f^{(3)}(z) = \sinh(z)$, etc. In general the even derivatives are $f^{(2k)}(z) = \cosh(z)$ for $k = 0, 1, 2, \ldots$, and the odd derivatives are $f^{(2k+1)}(z) = \sinh(z)$ for $k = 0, 1, 2, \ldots$,. Now evaluate these derivatives at $z = 0$ and get: $f^{(2k)}(0) = \cosh(0) = 1$ for $k = 0, 1, 2, \ldots$, $f^{(2k+1)}(0) = \sinh(0) = 0$ for $k = 0, 1, 2, \ldots$.

the coefficients of the Maclaurin series are

$a_{2k} = \frac{f^{(2k)}(0)}{(2k)!} = \frac{1}{(2k)!}$ for $k = 0, 1, 2, \ldots$,

$a_{2k+1} = \frac{f^{(2k+1)}(0)}{(2k+1)!} = 0$ for $k = 0, 1, 2, \ldots$. and the sequence of

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coefficients is

$$1, 0, \frac{1}{2!}, 0, \frac{1}{4!}, 0, \frac{1}{6!}, 0, \frac{1}{8!}, 0, \frac{1}{10!}, 0, \frac{1}{12!}, \ldots$$

Or if you prefer, you can write it as:

$$a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} \frac{1}{(2n)!}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

The series is usually expressed by adding up the non-zero even powers

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \frac{z^8}{8!} + \frac{z^{10}}{10!} + \frac{z^{12}}{12!} + \ldots$$

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k}$$

Or if you prefer the series can be written

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

as

$$= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} \frac{1}{n!} z^n$$

We are done.

Remark. If this last series looks strange, then recall

that $\cosh(z) = \frac{1}{2} e^z + \frac{1}{2} e^{-z}$, and

that $e^{-z} = \sum_{n=0}^{\infty} \frac{1}{n!} (-z)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} z^n$. Hence, we obtain

$$\cosh(z) = \frac{1}{2} e^z + \frac{1}{2} e^{-z}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \frac{1}{n!} z^n + \sum_{n=0}^{\infty} \frac{1}{2} (-1)^n \frac{1}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} \frac{1}{n!} z^n$$

We are really done.

1 (c). $\text{Log}(1+z)$.

Answer. $\text{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$ valid for $z \in D_1(0)$.

Solution. Given $f(z) = \mathrm{Log}(1+z)$, the derivatives are

$$f'(z) = \frac{1}{(1+z)}, \quad f^{(2)}(z) = -\frac{1!}{(1+z)^2}, \quad f^{(3)}(z) = \frac{2!}{(1+z)^3},$$

$$f^{(4)}(z) = \frac{3!}{(1+z)^4},$$ etc. In

general $f^{(n)}(z) = \dfrac{(-1)^{n-1}(n-1)!}{(1+z)^n}$ for $n = 1, 2, \ldots$. Evaluate the derivative at $z = 0$ and get $f(0) = \mathrm{Log}(1+0) = 0$, and

$$f^{(n)}(0) = \frac{(-1)^{n-1}(n-1)!}{(1+0)^n} = (-1)^{n-1}(n-1)!$$

the coefficients of the Maclaurin series are

$$a_n = \frac{f^{(2k)}(0)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$$ for $n = 1, 2, \ldots$, and

the sequence of coefficients is

$$0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{9}, -\frac{1}{10}, \frac{1}{11}, -\frac{1}{12}, \ldots$$

Hence, the Maclaurin series is

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \frac{z^{11}}{11} - \frac{z^{12}}{12} + \ldots$$

$$\mathrm{Log}(1+z) = 0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} z^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

We are done.

Exercise 2. Using methods other than computing derivatives, find the Maclaurin series for

2 (a). $(\cos(z))^3$.

Hint. Use the trigonometric identity $4(\cos(z))^3 = 3\cos(z) + \cos(3z)$.

Answer. $(\cos(z))^3 = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (3+9^n)}{4*(2n)!} z^{2n}$ valid for all z.

Notes

Solution. Use the known series for $\cos(z)$ which is valid for all z:

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

. Replace z with $3z$ and get the series for $\cos(3z)$:

$$\cos(3z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (3z)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} z^{2n}.$$

Using these series we obtain

$$(\cos(z))^3 = \frac{3}{4}\cos(z) + \frac{1}{4}\cos(3z)$$

$$= \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} z^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (3+9^n)}{4*(2n)!} z^{2n}$$

Therefore, $(\cos(z))^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (3+9^n)}{4*(2n)!} z^{2n}$ is valid for all z.

We are done.

Exercise 8. Suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an entire function.

8 (a). Find a series representation for $\overline{f(z)}$, using powers of $\overline{z}$.

Answer. $\overline{f(z)} = \sum_{n=0}^{\infty} \overline{c_n} (\overline{z})^n$.

Solution. Start with $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and conjugate each term in the series

$$\overline{f(z)} = \overline{\sum_{n=0}^{\infty} c_n z^n}$$

$$= \sum_{n=0}^{\infty} \overline{c_n z^n}$$

$$= \sum_{n=0}^{\infty} \overline{c_n} (\overline{z})^n$$

8 (b). Show that $\overline{f(\overline{z})}$ is an entire function.

Answer. $\overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{c_n} z^n$.

Solution. Substitute $\overline{z}$ into the series for $f(z)$ and get

$f(\overline{z}) = \sum_{n=0}^{\infty} c_n (\overline{z})^n$. Now take the conjugate and

$$\overline{f(\overline{z})} = \overline{\sum_{n=0}^{\infty} c_n (\overline{z})^n}$$

$$= \sum_{n=0}^{\infty} \overline{c_n}\, \overline{(\overline{z})^n}$$

$$= \sum_{n=0}^{\infty} \overline{c_n} (\overline{\overline{z}})^n$$

$$= \sum_{n=0}^{\infty} \overline{c_n} (z)^n$$

obtain $$= \sum_{n=0}^{\infty} \overline{c_n} z^n$$ Therefore, $\overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{c_n} z^n$ is valid for all z.

Now termwise differentiation can be used to obtain the derivative

$$\frac{d}{dz}\overline{f(\overline{z})} = \frac{d}{dz}\sum_{n=0}^{\infty} \overline{c_n} z^n$$

$$= \sum_{n=0}^{\infty} \overline{c_n} \frac{d}{dz} z^n$$

of $\overline{f(\overline{z})}$: $$= \sum_{n=1}^{\infty} \overline{c_n}\, n\, z^{n-1}$$ Therefore, $\overline{f(\overline{z})}$ is an entire function.

Exercise 9.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \ldots$,

where the coefficients c_n are the Fibonacci numbers defined by

$c_0 = 1,\ c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$, for $n \geq 2$.

9 (a). Show that $f(z) = \frac{1}{1 - z - z^2}$, for all $z \in D_R(0)$ for some number R.

Solution. Observe that

$$1 + z f(z) + z^2 f(z) = 1 + z\sum_{n=0}^{\infty} c_n z^n + z^2 \sum_{n=0}^{\infty} c_n z^n$$

$$= 1 + \sum_{n=0}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2}$$

Change the index of summation in the series and write it as follows

$$1 + z f(z) + z^2 f(z) = 1 + \sum_{n=0}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2}$$

$$= 1 + \sum_{n=1}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n$$

$$= 1 + c_0 z + \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n$$

$$= 1 + z + \sum_{n=2}^{\infty} (c_{n-1} + c_{n-2}) z^n$$

Now use the relation $c_n = c_{n-1} + c_{n-2}$ for $n \geq 2$ to get

$$1 + z f(z) + z^2 f(z) = 1 + z + \sum_{n=2}^{\infty} (c_{n-1} + c_{n-2}) z^n$$

$$= c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n$$

$$= \sum_{n=0}^{\infty} c_n z^n$$

$$= f(z)$$

Thus we have, $1 + z f(z) + z^2 f(z) = f(z)$.

Rearrange the terms, $f(z) - z f(z) - z^2 f(z) = 1$, and solve for $f(z)$.

Therefore, $f(z) = \dfrac{1}{1 - z - z^2}$, for all $z \in D_R(0)$ for some number R.

We are done.

7.7 SUMMARY

We study in this unit about M-test with its examples. We study also infinite series and uniqueness of infinite series. We study Taylor series expansion and its general representation with examples. We study uniform convergence series.

7.8 KEYWORD

Expansion : The political strategy of extending a state's territory by encroaching on that of other nations

Conjugate : Give the different forms of (a verb in an inflected language such as Latin) as they vary according to voice, mood, tense, number, and person

Uniform : Remaining the same in all cases and at all times; unchanging in form or character

7.9 QUESTIONS FOR REVIEW

1. Find a sequence a_n of real numbers such that $\sum a_n$ converges but $\prod(1+a_n)$ diverges.

2. Find a sequence a_n of real numbers such that $\sum a_n$ diverges but $\prod(1+a_n)$ converges (and is greater than zero).

3. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \ldots$,

where the coefficients c_n are the Fibonacci numbers defined by

$c_0 = 1,\ c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$, for $n \ge 2$.

4 Suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an entire function. Find a series representation for $\overline{f(z)}$, using powers of $\overline{z}$.

5 For each r, (where $0 < r < 1$), the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly on the closed disk $\overline{D_r}(0) = \{z : |z| \le r\}$.

7.10 SUGGESTION READING AND REFERENCES

Notes

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7.11 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 4

2 Check in Section 3

Check In Progress-II

Answer Q. 1 Check in section 5

2 Check in Section 5